Reconstruction of 1-D and 2-D Finite Length Signals from Partial Information

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Abstract. In many applications, a finite 1-D or 2-D signal is to be reconstructed from partial information. Common situations include missing elements of the sequence itself (i.e subsampling) and incomplete information about its Discrete Fourier transform phase or magnitude. In general, these are ill-posed inverse problems that require regularizing assumptions to solve. This paper addresses combinations of constraints on a finite signal that allow exact reconstruction with incomplete knowledge of Fourier phase and missing samples. Some numerical results are also shown.

1. Introduction

We know that there exists a one-one correspondence between a finite length signal and its Discrete Fourier transform (DFT). In fact the DFT is a 1-1, onto, linear map. In many applications however, some of the Fourier domain (also called spectral) information is not known. For example, the problem of restoring a signal from its DFT magnitude alone [1] arises in a variety of different contexts [**3**]. Without additional information however this is an ill-posed problem, since for a discrete signal of length N, we have 2^{N-1} such sequences with the same magnitude. The goal is therefore to find additional constraints or information on the signal [**4**] to guarantee a unique solution. In this paper we show that under some fairly general conditions it is possible to recover a signal from its Fourier domain information and also from a sub-sampled portion of the original signal.

Definition 1. *The Fourier transform of a finite signal $(x[n])_{n=0}^{N-1}$ of length N is defined as:*

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n} \tag{1}$$

As a complex valued function $X(e^{j\omega})$ can be written in polar form as :

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\phi_x(\omega)} \tag{2}$$

where $|X(e^{j\omega})|$ and $\phi_x(\omega)$ are called the Fourier magnitude and Fourier phase of $(x[n])$ respectively.

Definition 2. *The discrete Fourier transform(DFT) of $x[n]$ also called the N-point DFT is calculated by evaluating the Fourier transform at N equally spaced points on the unit circle $|z| = 1$:*

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \tag{3}$$

and in polar form as :

$$X[k] = |X[k]|e^{j\phi_x(k)}$$

for $k = 0, 1, \cdots, N-1$

Given $(X[k])_{k=0}^{N-1}$, the sequence $(x[n])$ can be derived by the Inverse Discrete Fourier transform (IDFT):

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi nk/N}$$

Definition 3. *Associated with a finite signal x[n] is an autocorrelation sequence $C_x = \{c_{-(N-1)}, \cdots, c_{N-1}\}$ defined by :*

$$c_j = \begin{cases} \sum_{k=0}^{N-j-1} x[k]x[k+j]^* & : \quad j = 0,1,...,N-1 \\ c_{-j}^* & : \quad j < 0 \end{cases} \tag{4}$$

In this expression, the asterisk denotes complex conjugation

[1] generally referred to as the phase retrieval problem

DEFINITION 4 (Magnitude Spectrum). *The Fourier transform of* C_x *is* $|X(k)|^2$*; i.e*

$$C[k] = |X[k]|^2 = \sum_{i=-(N-1)}^{N-1} c_k e^{-j2\pi ki/N} \tag{5}$$

This implies that the autocorrelation sequence C_x can be obtained from the magnitude spectrum $|X(k)|$ by the inverse Fourier transform. Because of this relation, knowledge of a signals magnitude spectrum and its autocorrelation sequence are equivalent. Given a finite signal x[n], C_x is well defined by Definition 3. Clearly, however, not every sequence of 2n+1 numbers arises as the autocorrelation sequence of some signal x of length n+1. Two (equivalent) and necessary conditions for a sequence $C = \{c_{-n}, \cdots, c_n\}$ to be the autocorrelation of some signal $\{x[0], \cdots, x[n]\}$ are :

1. The Fourier transform of C, as written in Definition 1 is non-negative
2. The sequence formed by C_x is positive definite

2. Reconstruction from Discrete Fourier Magnitude and some samples

This section of the paper addresses the problem of reconstructing a finite discrete signal given its Fourier Transform magnitude and some samples. In particular we relax certain conditions under which uniqueness can be obtained and extend the work of [**1**] ,[**2**] and [**4**] . We also show how the work can be extended to 2-dimensions .

2.1. Uniqueness theorems for 1-D signals. Because of the relationship that a signals's spectral magnitude being equivalent to knowing its autocorrelation sequence, if we assume our signal to be real and ignoring the redundant equations we are reduced to solving a system of N equations in N unknowns. These equations come from matching powers of $e^{\frac{i2\pi k}{N}}$ in

$$\overline{X[k]}X[k] = C[k]$$

Since we assume our signal x[n] to be real, N of the equations become redundant and we can rewrite the rest explicitly in matrix form :

$$\begin{pmatrix} x[0] & x[1] & x[2] & \dots & x[N-1] \\ 0 & x[0] & x[1] & \dots & x[N-2] \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & x[1] \\ 0 & 0 & 0 & \dots & x[0] \end{pmatrix} \begin{pmatrix} x[0] \\ x[1] \\ \dots \\ x[N-2] \\ x[N-1] \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \\ \dots \\ c_{N-2} \\ c_{N-1} \end{pmatrix} \tag{6}$$

Note that the system is not linear but homogeneous of degree 2. We investigate the conditions under which one could reduce this system to a set of linear equations by knowing certain samples.

Below are some new results about reconstructing a signal from its autocorrelation sequence and some samples. The proofs are discussed in the appendix.

THEOREM 2.1. *Let* $x[n]$ *be a real discrete signal which is zero outside the interval* $0 \le n \le N-1$ *and* $C_x = \{c_0, c_1, \cdots, c_{N-1}\}$ *be its corresponding autocorrelation sequence. Then the entire sequence* $x[n]$ *can be uniquely recovered from* C_x *and* $m = \frac{N}{2}$ *samples provided for* $1 \le k \le \frac{N}{2}$ *(Assume N even):*

1. $p = \frac{N}{k} = \frac{N}{k}$ *and p is even.*
2. *The samples* $\{x[jk], x[jk+1], \cdots, x[(j+1)k-1]\}$ *are known for* $j = 0, 2, \cdots, p$ *i.e we have k successive known samples followed by k successive unknown samples of the sequence* $x[n]$
3. $x[0] \neq 0$ *and* $x[N-1] \neq 0$

THEOREM 2.2. *Let* $x[n]$ *be a real discrete signal which is zero outside the interval* $0 \le n \le N-1$ *and* $C_x = \{c_0, c_1, \cdots, c_{N-1}\}$ *be its corresponding autocorrelation sequence. Then the entire sequence* $x[n]$ *can be uniquely recovered from* C_x *and* $m = \frac{N}{3}$ *samples provided for* $1 \le k \le \frac{N}{3}$ *(Assume N divisible by 3):*

1. $p = \frac{N}{3k} = \frac{N}{3k}$ *and p is even or p=1*
2. *The following* $kp(=\frac{N}{3})$ *samples are known* $(p \neq 1)$*:*
   $x[j_1k], x[j_1k+1], \cdots, x[(j_1+1)k-1]$ *for* $j_1 = 0, 2, \cdots, \frac{p-2}{2}$ *and*
   $x[N-j_2k], x[N-j_2k+1], \cdots, x[N-(j_2-1)k-1]$ *for* $j_2 = 2, 4, \cdots, p$

2.2. Extension to 2-D Sequences. The results from the previous section for discrete 1-D signal reconstruction problem can be easily extended to the 2-D case. We assume that the support of the 2-D signal (image) is a square [0,N-1] x [0,N-1]. As seen previously in the 1-D case we can rewrite the non-linear system of equations (assuming $x[m,n]$ is real) in terms of its 2-D autocorrelation defined as:

$$c[k,l] = \sum_{i=0}^{N-1-k} \sum_{j=0}^{N-1-l} x[i,j]x[i+k,j+l] \tag{7}$$

as follows :

$$\begin{pmatrix} A_0 & A_1 & A_2 & \cdots & A_{N-1} \\ A_1 & A_2 & A_3 & \cdots & 0 \\ A_2 & A_3 & A_4 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{N-2} & A_{N-1} & 0 & \cdots & 0 \\ A_{N-1} & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \overline{x_0} \\ \overline{x_1} \\ \overline{x_2} \\ \cdots \\ \overline{x_{N-2}} \\ \overline{x_{N-1}} \end{pmatrix} = \begin{pmatrix} \overline{c_0} \\ \overline{c_1} \\ \overline{c_2} \\ \cdots \\ \overline{c_{N-2}} \\ \overline{c_{N-1}} \end{pmatrix} \tag{8}$$

which is block upper triangular system of N^2 equations. Here each $A_k, k = 0, 1, \cdots, N-1$ is an upper triangular N by N matrix defined as follows:

$$A_k = \begin{pmatrix} x[k,0] & x[k,1] & x[k,2] & \cdots & x[k,N-1] \\ x[k,1] & x[k,2] & x[k,3] & \cdots & 0 \\ x[k,2] & x[k,3] & x[k,4] & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x[k,N-2] & x[k,N-1] & 0 & \cdots & 0 \\ x[k,N-1] & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{9}$$

and the $\overline{x_k}$ and $\overline{c_k}$ are N by 1 vectors defined as follows:

$$\overline{x_k}^T = [x[k,0], x[k,1], \cdots, x[k,N-1]]$$
$$\overline{c_k}^T = [c[k,0], c[k,1], \cdots, c[k,N-1]]$$

THEOREM 2.3. *Let $x[n_1, n_2]$ be a real discrete 2-d signal which is zero outside the square region $R_x = \{(n_1, n_2) : 0 \le n_1 \le N-1, 0 \le n_2 \le N-1\}$ and $C_x = \{c_{0,0}, \cdots, c_{N-1,N-1}\}$ be its corresponding autocorrelation sequence. Then the entire sequence $x[n_1, n_2]$ can be uniquely recovered from C_x and $m = \frac{N^2}{2}$ samples provided :*

1. *$N = 2^l$ for l any positive integer*
2. *The m known samples are $\{\overline{x_{jk}}, \cdots, \overline{x_{(j+1)k}}\ \}$ for $j = 1, 3, \cdots, p-1$ where $p = \lfloor \frac{N}{k} \rfloor$ for $1 \le k \le \frac{N}{2}$*
3. *The samples x[0,0] and x[N-1,N-1] are non zero.*

THEOREM 2.4. *Let $x[n_1, n_2]$ be a real discrete 2-d signal which is zero outside the square region $R_x = \{(n_1, n_2) : 0 \le n_1 \le N-1, 0 \le n_2 \le N-1\}$ and $C_x = \{c_{0,0}, \cdots, c_{N-1,N-1}\}$ be its corresponding autocorrelation sequence. Then the entire sequence $x[n_1, n_2]$ can be uniquely recovered from C_x and $m = \frac{N^2}{3}$ samples provided :*

1. *$N = 3^l$ for l any positive integer*
2. *The m known samples are $\overline{x_j}, \cdots, \overline{x_{j+k-1}}$ for $j = 2, 5, \cdots, pk-1$ where $p = \frac{N}{3k}$ for $1 \le k \le \frac{N}{3}$*
3. *The samples x[0,0] and x[N-1,N-1] are non zero*

The proof for the above two theorems follow similar lines as the 1-D case.

2.3. Numerical Results. The results of theorem 2.2 are tested with a 1-D chirp signal of length 1020 and a square image of support 18. Here $k = \frac{N}{6}$. The relative error calculated is defined as follows:

$$error[i] = \frac{|x[i] - xr[i]|}{|x[i]|} \tag{10}$$

for the 1-D case and

$$error[i,j] = \frac{|x[i,j] - xr[i,j]|}{|x[i,j]|} \tag{11}$$

for the 2-D case, where x is the true value and xr is the reconstructed value. The results along with the relative error plots are shown in Figures 1-8. Addition of noise into the data may or may not cause large errors, depending explicitly on the values corrupted and how this effects the systems involved. A sensitivity error analysis is currently ongoing.

3. Reconstruction from Sub-Sampling

DEFINITION 5. *Let $A_0 = \{a_0, a_1, a_2 \cdots a_{N-1}\}$ be a sequence of length N with real coefficients where $N = 2^k$ (k being an integer). We now define k new sequences from A_0 as follows :*
$A_1 = \{a_0, a_2, a_4, \cdots, a_{N-2^1}\}; N_1 = 2^{k-1}$
$A_2 = \{a_0, a_4, a_8, \cdots, a_{N-2^2}\}; N_2 = 2^{k-2}$

$\vdots$

$A_j = \{a_0, a_{2^j}, a_{2^{j+1}}, \cdots, a_{N-2^j}\}; N_j = 2^{k-j}$

$\vdots$

$A_k = \{a_0\}; N_k = 2^{k-k} = 1$

LEMMA 3.1. *If A_0 is a sequence of length N and $A_1, \cdots, A_k$ are sequences as defined above, then we can reconstruct the terms of A_0 provided we know the DFT magnitude of A_j for $j = 1, 2, \cdots, k$ and the sign of a_0*

DEFINITION 6. *Let $A_0 = \{a_0, a_1, a_2, \cdots, a_{N-1}\}$ be a sequence of length N with real coefficients where $N = 3^k$ (k being an integer). We now define k new sequences from A_0 as follows :*
$A_1 = \{a_0, a_3, a_6, \cdots, a_{N-3^1}\}; N_1 = 3^{k-1}$
$A_2 = \{a_0, a_9, a_{18}, \cdots, a_{N-3^2}\}; N_2 = 3^{k-2}$
$\vdots$
$A_j = \{a_0, a_{3^j}, a_{3^{j+1}}, \cdots, a_{N-3^j}\}; N_j = 3^{k-j}$
$\vdots$
$A_k = \{a_0\}; N_k = 3^{k-k} = 1$

LEMMA 3.2. *If A_0 is a sequence of length N and $A_1, \cdots, A_k$ are sequences as defined in definition 2 above, then we can reconstruct the terms of A_0 provided we know the DFT magnitude of the sequences A_j for $j = 1, 2, \cdots, k$*

DEFINITION 7. *Let $A_0 = \{a_0, a_1, a_2, \cdots, a_{N-1}\}$ be a sequence of length N with real coefficients where $N = 2^k$ (k being an integer). We now define k new sequences from A_0 as follows :*
$A_1 = \{a_0, a_1, a_2 \cdots, a_{N_1-1}\}; N_1 = 2^{k-1}$
$A_2 = \{a_0, a_1, a_2 \cdots, a_{N_2-1}\}; N_2 = 2^{k-2}$
$\vdots$
$A_j = \{a_0, a_1, a_2, \cdots, a_{N_j-1}\}; N_j = 2^{k-j}$
$\vdots$
$A_k = \{a_0\}; N_k = 2^{k-k} = 1$

LEMMA 3.3. *If A_0 is a sequence of length N and $A_1, \cdots, A_k$ are sequences as defined in definition 3 above, then we can reconstruct the terms of A_0 provided we know the DFT magnitude of the sequences A_j $j = 1, 2, \cdots, k$ and the sign of a_0*

DEFINITION 8. *Let $A_0 = \{a_0, a_1, a_2, \cdots, a_{N-1}\}$ be a sequence of length N with real coefficients where $N = 3^k$ (k being an integer). We now define k new sequences from A_0 as follows :*
$A_1 = \{a_0, a_1, a_2, \cdots, a_{N_1-1}\}; N_1 = 3^{k-1}$
$A_2 = \{a_0, a_1, a_2, \cdots, a_{N_2-1}\}; N_2 = 3^{k-2}$
$\vdots$
$A_j = \{a_0, a_1, a_2, \cdots, a_{N_j-1}\}; N_j = 3^{k-j}$
$\vdots$
$A_k = \{a_0\}; N_k = 3^{k-k} = 1$

LEMMA 3.4. *If A_0 is a sequence of length N and $A_1, \cdots, A_k$ are sequences as defined in definition 4 above, then we can reconstruct the terms of A_0 provided we know the DFT magnitude of the sequences A_j $j = 1, 2, \cdots, k$ and the sign of a_0*

4. Conclusions

In this paper we talk about exact reconstruction of finite 1-D and 2-D signals from partial Fourier information and some samples. The partial Fourier information used here have been in the form of the Discrete Fourier transform magnitude. Numerically it can be shown the the reconstruction is fairly successful provided the linear systems used are not badly conditioned. We also show some results related to reconstruction from sub-sampled versions of the original signal which could have some applications in wavelet analysis. Current research involves reconstruction of signals from phase information and also signals with ± 1 terms from phase and magnitude information.

Appendix A. Proof of Theorem 2.2

- CASE A: p=1 This is the case where the first $\frac{N}{3}$ samples of the sequence are known. Using these samples and the last $\frac{N}{3}$ autocorrelation coefficients we can easily solve for the last $\frac{N}{3}$ samples of the sequence. The problem now reduces to solving the middle $\frac{N}{3}$ samples of the sequence. For that we use the equations arising from the autocorrelation coefficients $c_p, c_{p+1}, \cdots, c_k$:

$$\begin{pmatrix} x[0] & x[1] & \dots & x[k-1] & x[k] \\ 0 & x[0] & \dots & x[k-2] & x[k-1] \\ 0 & 0 & \dots & x[k-3] & x[k-2] \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & x[p-1] & x[p] \\ 0 & 0 & \dots & x[p-2] & x[p-1] \end{pmatrix} \begin{pmatrix} x[p] \\ x[p+1] \\ x[p+2] \\ \dots \\ \dots \\ x[N-2] \\ x[N-1] \end{pmatrix} = \begin{pmatrix} c_p \\ c_{p+1} \\ c_{p+2} \\ \dots \\ \dots \\ c_{k-1} \\ c_k \end{pmatrix} \quad (12)$$

where $k = \frac{2N}{3} - 1$ and $p = \frac{k+1}{2}$.

This can be rearranged and written in matrix form as:

$$A\overline{x_u} = \overline{c} \quad (13)$$

where :

(14)

$$A = \begin{pmatrix} x[0]+x[k+1] & x[1]+x[k+2] & \dots & x[p-1]+x[N-1] \\ x[k+2] & x[0]+x[k+3] & \dots & x[p-2] \\ x[k+3] & x[k+4] & \dots & x[p-3] \\ \dots & \dots & \dots & \dots \\ x[N-2] & x[N-1] & \dots & x[0] \\ x[N-1] & 0 & \dots & x[0] \end{pmatrix}, \overline{x_u} = \begin{pmatrix} x[p] \\ x[p+1] \\ x[p+2] \\ \dots \\ x[k-1] \\ x[k] \end{pmatrix}, \overline{c} = \begin{pmatrix} \hat{c_0} \\ \hat{c_1} \\ \hat{c_2} \\ \dots \\ c_{\frac{N}{3}-2}\hat{} \\ c_{\frac{N}{3}-1}\hat{} \end{pmatrix}$$

Here $\hat{c_i} = c_{\frac{N}{3}+i} - S_i$ and $S_{\frac{N}{3}-k} = \sum_{j=k}^{\frac{N}{3}-1} x[j]x[j+\frac{2N}{3}-k]$. This system has a solution provided A^{-1} exists i.e it is non-singular.

- CASE B: p is even
  The proof of this follows from CASE A above except that we are now dealing with smaller subsystems of equations, but because of the structure of the known and unknown samples from the hypotheses of the theorem we are always able to form (in a recursive way) sub-systems of equations which are always linear in terms of unknown samples like in CASE A.

Appendix B. Proof of lemma 3.1

We will use the method of induction to prove the above lemma. From the hypothesis of the lemma we know a_0. Noting that information about the Discrete Fourier transform magnitude of the sequence $A_j$is equivalent of knowing its autocorrelation we have the following :
From a_0 and c_i^{k-1}, the autocorrelation of the coefficients of A_{k-1}, we can easily solve for a_{2^k-1} and hence recover A_{k-1}. We will show that given A_j and its Discrete Fourier transform magnitude we can reconstruct A_{j-1}. Since the autocorrelation of the coefficients of A_{j-1} i.e c_i^{j-1} are known we can solve for the unknown coefficients of A_{j-1} from the following linear system:

$$\begin{pmatrix} a_0 + a_{2^j} & a_{2^j} + a_{2^j+1} & \dots & a_{N-2^j} \\ a_{2^j+1} & a_0 + a_{6*2^j-1} & \dots & a_{N-2^j-2} \\ a_{6*2^j-1} & a_{2^j+2} & \dots & a_{N-2^j-4} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_0 \end{pmatrix} \begin{pmatrix} a_{2^j-1} \\ a_{3*2^j-1} \\ a_{5*2^j-1} \\ \dots \\ a_{N-2^j} \end{pmatrix} = \begin{pmatrix} c_1^{j-1} \\ c_3^{j-1} \\ c_5^{j-1} \\ \dots \\ c_{N_j-1}^{j-1} \end{pmatrix}$$

The proofs of lemma 3.2,3.3, and 3.4 are similar to the proof of lemma 3.1 and follow directly from theorem 2.1 and theorem 2.2 .

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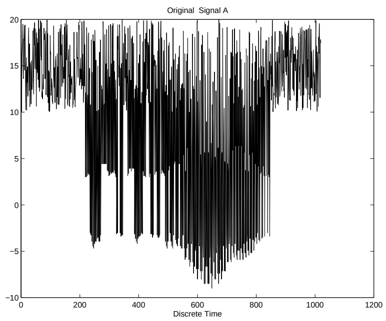


FIGURE 1

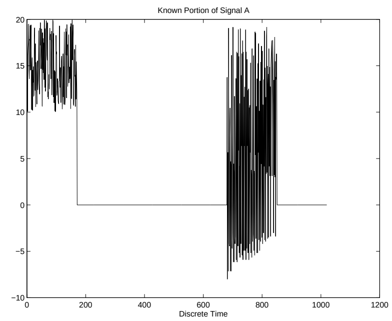
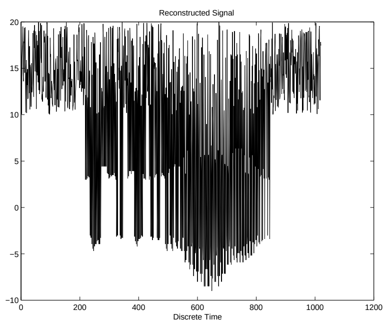


FIGURE 2



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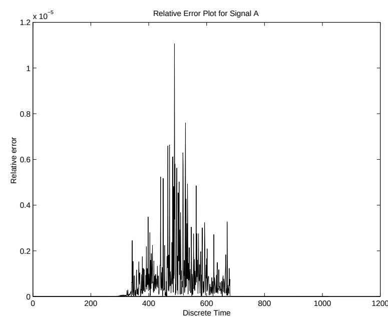

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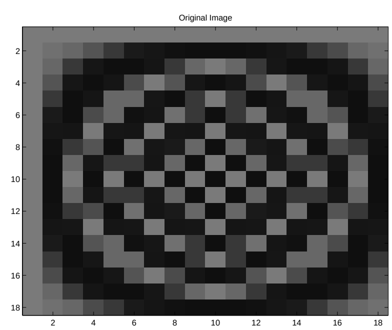

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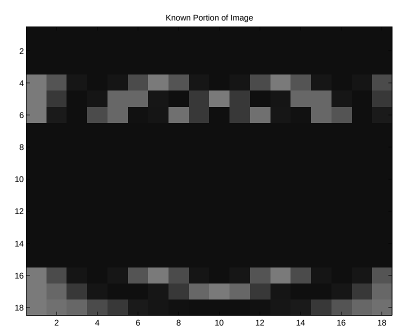

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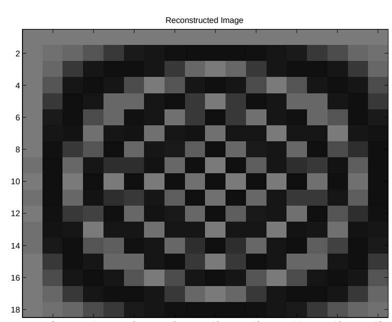

FIGURE 7

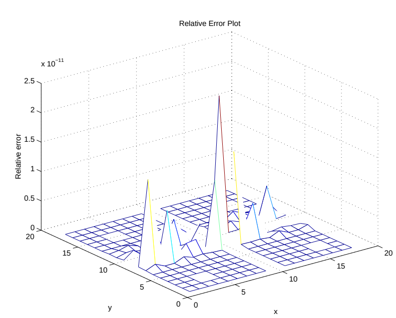

FIGURE 8