Improved Wavelet Based Estimations of Nearly $1/f$ Processes

S. Wada and N. Ito

Department of Electrical Engineering, Tokyo Denki University
E-mail: wada@cck.dendai.ac.jp

Abstract - Nearly $1/f$ processes are known as an important stochastic model for scale invariant data analysis in a number of fields. In this paper, first a concrete generation method of nearly $1/f$ processes based on a realization of non-integer integral computing elements model is presented. Next, two effective estimation methods of nearly $1/f$ processes using a wavelet with complementary sampling and a wavelet packet based EM algorithm are proposed. It is shown that the wavelet with complementary sampling based estimation gives more reliable parameter estimation values even when the data length is short, and the flexible wavelet packet based estimation gives more precise parameter values for a wide range of generated nearly $1/f$ processes in comparison with the wavelet transform based estimation.

I. Introduction

The $1/f$ processes are known as an important stochastic model for scale invariant data analysis in a number of fields such as geophysical and economic data, biological signals and computer network traffic analyses [1]–[4]. The $1/f$ processes are closely related to so-called fractal processes having a number of interesting characteristics. A parameter of $1/f$ processes as well as the fractal dimension is used for distinguishing among various man-made and natural objects [1],[4]. The parameter estimation of observed statistical fractal signals is an important subject. Generating fractal signals with desired parameters is also important subject in a number of application areas. The $1/f$ processes or fractal processes can be analyzed and represented in the framework of wavelet theory [1]–[3]. In [1], the notation of $1/f$ processes is extended to nearly $1/f$ processes suitable for real signal analysis and modeling. Further, an effective estimation method of $1/f$ processes embedded in white background noise using a wavelet transform and EM algorithm is proposed.

In this paper, first a concrete generation method of nearly $1/f$ processes based on a realization of non-integer integral computing elements model is presented. The nearly $1/f$ processes are generated as an output signal of the non-integer integral computing elements model for a real white input signal. Next, two effective estimation methods of the nearly $1/f$ processes using a wavelet with complementary sampling and a wavelet packet based EM algorithm are shown. The wavelet packet decomposition for nearly $1/f$ processes is obtained by constructing a discrete wavelet packet tree according to a minimum entropy criteria. Then, an EM algorithm is applied to estimate parameters of nearly $1/f$ processes. In the simulations, it is shown that the wavelet with complementary sampling gives more reliable estimation results even when the observed data length is short. Further, it is shown that there exists a limitation with respect to estimation range in the wavelet transform based estimation. The flexible wavelet packet based estimation can improve parameters estimation in a wide range of nearly $1/f$ processes.

II. Generation of Nearly $1/f$ Processes

In this section, a generation method of nearly $1/f$ processes based on a realization of non-integer integral computing elements model is presented. Now, the definition of nearly $1/f$ processes is introduced [1].
Definition: The nearly $1/f$ processes are defined as having power spectra bounded according to

$$\frac{k_1}{|\omega|^\gamma} \leq S(\omega) \leq \frac{k_2}{|\omega|^\gamma} \qquad (1)$$

where k_1 and k_2 satisfy $0 \leq k_1 \leq k_2 < \infty$ but are otherwise arbitrary.

The fractal dimension D can be computed directly with the parameter γ by

$$D = \frac{5-\gamma}{2}. \qquad (2)$$

In order to obtain a signal model of nearly $1/f$ processes, an ARMA model approach is known. To identify the ARMA system, complex computation is necessary to determine the coefficients using observed nearly $1/f$ signals. The nearly $1/f$ processes can be modeled as an output signal of a non-integer integral computing elements model for a real white input signal. The ideal non-integer integral computing element is a non-causal system whose transfer function is represented by

$$F(s) = \frac{1}{s^p} \quad (0 \leq p \leq 1). \qquad (3)$$

Thus, it is approximated by the following rational transfer function:

$$F(s) \simeq \frac{\sqrt{\prod_{i=1}^{n_1}(1+\delta_i^2)\prod_{i=1}^{n_2}(s+\sigma_i)}}{\sqrt{\prod_{i=1}^{n_2}(1+\sigma_i^2)\prod_{i=1}^{n_1}(s+\delta_i)}} \quad (0 \le p \le 1). \tag{4}$$

Here the poles and zeros are calculated with the Jacobi's elliptic function expressed by

$$\delta_i = \sqrt{k'}\mathrm{tn}\left(\frac{2i-1-p}{n}K, k\right), \tag{5}$$

$$\sigma_i = \sqrt{k'}\mathrm{tn}\left(\frac{2i-1+p}{n}K, k\right), \tag{6}$$

where

$$\begin{cases} n_1 = \dfrac{n+1}{2}, \ n_2 = \dfrac{n-1}{2} & (n : \mathrm{odd}) \\ n_1 = n_2 = \dfrac{n}{2} & (n : \mathrm{even}) \end{cases}$$

The efficient algorithm to calculate the poles and zeros represented by (5) and (6) is shown in [5]. By selecting a few parameters such as k' and n, the bandwidth and ripples of frequency characteristics of non-integer integral computing elements can be easily controlled. Then, discrete-time recursive filter systems for the non-integer integral computing elements model are obtained by transforming the transfer function represented by (4).

III. Improved Wavelet Based Parameters Estimation with EM Algorithm

In this section, two parameter estimation methods using a wavelet with complementary sampling and a wavelet packet based EM algorithm are proposed.

A. Parameters Estimation Based on Wavelet [1]

First, the wavelet representation of $1/f$ process based on Karhunen-Loeve-like expansion is introduced[2]. An observed signal $r(t)$ is assumed to be represented by

$$r(t) = x(t) + w_n(t), \quad -\infty < t < \infty \tag{7}$$

where $x(t)$ represents a Gaussian nearly $1/f$ process and $w_n(t)$ represents a zero-mean additive stationary white Gaussian background noise. Then, the wavelet coefficients of the signal $r(t)$ is represented by

$$r_n^m = x_n^m + w_n^m \tag{8}$$

where x_n^m and w_n^m represent wavelet coefficients of $x(t)$ and $w_n(t)$, respectively. The variance of the wavelet coefficients of $1/f$ processes is represented by

$$r_n^m = \sigma_m^2 = \sigma^2\beta^{-m} + \sigma_w^2 \tag{9}$$

$$\beta = 2^\gamma. \tag{10}$$

The estimation problem can be formulated to estimate a set of parameters

$$\Theta = (\beta, \sigma^2, \sigma_w^2). \tag{11}$$

The likelihood function expressed by

$$\mathcal{L}(\Theta) = \prod_{m,n\in\mathcal{R}} \frac{1}{\sqrt{2\pi\sigma_m^2}} \exp\left[-\frac{(r_n^m)^2}{2\sigma_m^2}\right] \tag{12}$$

is used in the estimation problem. The EM algorithm is applied to maximize the likelihood function using wavelet coefficients in the previous work [1], [6]. The EM algorithm is summarized as the following estimation and maximization steps:

E-step: Compute

$$U(\Theta, \hat{\Theta}^{(l)}).$$

M-step:

$$\max_{\Theta} U(\Theta, \hat{\Theta}^{(l)}) \rightarrow \hat{\Theta}^{(l+1)}.$$

The E-step and M-step are iterated alternately until the parameters converge certain values. It is noted that when $\sigma_w^2 = 0$ (or assumed), the parameters can be estimated directly, and the maximization process is simplified [1].

B. Wavelet with Complementary Sampling

The performance for the wavelet based EM algorithm mainly depends on the data length, the step of wavelet decomposition and the range of estimated γ. When the observed data length is sufficiently long and the large step of wavelet tree is realized, the parameter γ around 1 can be precisely estimated. However, when the observed data length is short (data length < 1000), the estimation result is not satisfactory. The average of the estimated γ values have large error, and the variance becomes large.

In order to reduce the estimation errors, the following two methods are proposed. The complementary sampling shown in Figure 1 is introduced in the wavelet tree decomposition using discrete-time filter bank systems. Using the complementary sampling scheme, we can obtain the necessary number of wavelet coefficients even if the data length is short. Although a number of calculation is necessary to obtain wavelet coefficients compared with the conventional wavelet decomposition, the estimation result is excellently improved as shown in the simulation. Now the concrete calculation methods for the improvement are shown.

[Method 1]

Using a set of wavelet coefficients $\{r_{n(i)}^m, i = 1, 2, \cdots, 2^M\}$ obtained by the complementary sampling, estimate $\{\hat{\beta}_{(i)}, i = 1, 2, \cdots, 2^M\}$ by computing

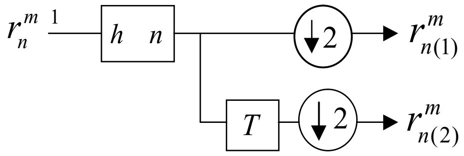


Figure 1: Complementary sampling of wavelet coefficients. $h[-n]$ represents the impulse response of the LPF for a discrete-time filter bank system.

2^M EM algorithm and calculate the following β.

$$\hat{\beta}_{m1} = \frac{1}{2^M} \sum_{i=1}^{2^M} \hat{\beta}_{(i)}^{l+1}.$$

[Method 2]

Using a set of wavelet coefficients $\{r_{n(i)}^m, i = 1, 2, \cdots, 2^M\}$ obtained by the complementary sampling, compute $\{\hat{\sigma}_{(i)}^2, i = 1, 2, \cdots, 2^M\}$ and take the average as follows:

$$\hat{\sigma}_{m,m2}^2 = \frac{1}{2^M N(m)} \sum_{i=1}^{2^M} \sum_{n \in \mathcal{N}} (r_{n(i)}^m)^2.$$

The method 1 or method 2 can be usefully applied in the wavelet based EM algorithm.

C. Wavelet Packet Tree Decomposition

Next approach is to estimate the parameter γ of nearly $1/f$ processes by using a wavelet packet with EM algorithm.

The wavelet packet is an expansion of a signal using a set of bases

$$\psi_n^{m,p}(t) = 2^{m/2} \psi^p (2^m t - n).$$

The coefficients are practically calculated using the observed data by iterated structure of discrete-time filter bank systems. In order to fit nearly $1/f$ processes, the discrete wavelet packet tree is constructed by minimizing the entropy function of wavelet packet coefficients [7]. The entropy function is shown by

$$E(p) = -\sum_i p_i^2 \log(p_i^2),$$

$$p_i = \frac{|x_i|^2}{\|x_i\|^2}$$

where x_i represents coefficients. It is noted that the wavelet packet tree structure strongly depends on analyzed nearly $1/f$ processes.

The M-step for the wavelet packet tree is represented by

$$\hat{\beta}_{ML} \leftarrow \sum_{m \in \mathcal{M}} C_m^p N_p(m) \hat{\sigma}_m^{p2} \beta^m = 0, \tag{13}$$

$$\sigma_{ML}^2 = \frac{\sum_{m \in \mathcal{M}} N_p(m) \hat{\sigma}_m^{p2} [\hat{\beta}_{ML}]^m}{\sum_{m \in \mathcal{M}} N_p(m)}. \tag{14}$$

Here

$$\hat{\sigma}_m^{p2} = \frac{1}{N_p(m)} \sum_{p \in \mathcal{P}} \sum_{n \in \mathcal{N}} (r_n^{m,p})^2, \tag{15}$$

$$C_m^p = \begin{cases} \dfrac{m}{\sum_{m \in \mathcal{M}} m N_p(m)} - \dfrac{1}{\sum_{m \in \mathcal{M}} N_p(m)} & (p(m) \neq 0) \\ 0 & (p(m) = 0) \end{cases},$$

$$\mathcal{P} = \{1, 2, \cdots, p(m)\},$$

$$N_p(m) = p(m) \cdot N(m)$$

and $p(m)$ represents the total number of wavelet packets at scale m.

IV. Simulations

Finally, simulation results for synthesis and analysis of nearly $1/f$ processes are shown to demonstrate the effectiveness of our methods.

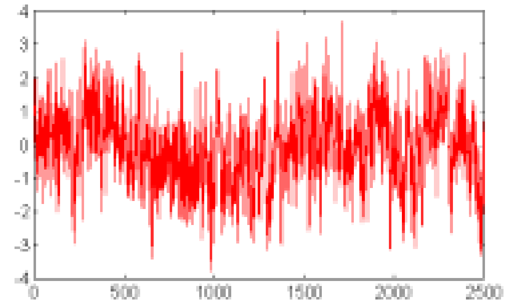
A. Wavelet with Complementary Sampling Based Estimation

Figure 2 shows a few examples for generation of nearly $1/f$ processes. 5-th order non-integer integral computing elements were designed with the parameters given by $\gamma = 0.8, 1.6$ and 2.4. It is shown that when the γ increases, the waveform becomes large and smooth.

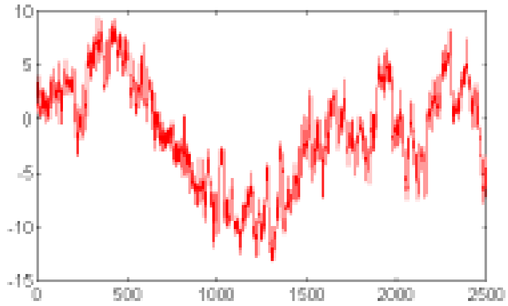
Using the generated nearly $1/f$ processes, the parameter γ is estimated based on the wavelet transform, method 1 and method 2. The results are shown in Table 1. The estimated γ by the method 1 shows smaller error compared with that of the conventional wavelet transform based method. However, a number of computation with EM algorithm is necessary in the method 1. On the other hand, the computation for the method 2 is almost equal to that of the wavelet transform. The comparative improvement of the estimation is achieved in the method 2. However, the effective range of the estimated γ is restricted $1 < \gamma < 1.6$ in the simulation.

	γ	Wavelet	Method 1	Method 2
Ave	0.8	0.77597	0.78481	0.86445
	1.2	1.16834	1.17676	1.24044
	1.6	1.56154	1.56712	1.57842
Er (%)	0.8	-3.0035	-1.8985	8.0559
	1.2	-2.6383	-1.9367	3.3703
	1.6	-2.4034	-2.0550	-1.3490
Var	0.8	0.08665	0.07870	0.08420
	1.2	0.09223	0.08389	0.07621
	1.6	0.09766	0.09190	0.05697

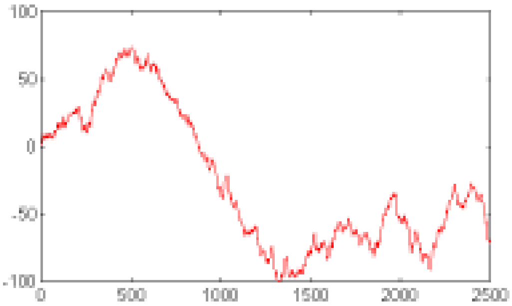
Table 1. Average, error ratio and variance for estimated γ values (Data length:512, WT step:7, Sample number:150)



(a)Nearly $1/f$ process ($\gamma = 0.8$).



(b)Nearly $1/f$ process ($\gamma = 1.6$).



(c)Nearly $1/f$ process ($\gamma = 2.4$).

Figure 2: Generation of nearly $1/f$ processes using non-integer integral computing elements model.

B. Wavelet Packet Based Estimation

Figure 3 shows a comparison of estimated γ using a discrete wavelet transform decomposition and a discrete wavelet packet decomposition. The data length is $2^{17} = 131072$. Daubechies 5-th order analyzing wavelet was used. It is shown that γ is accurately estimated around 1 in the wavelet transform. However, for larger γ, the estimation error becomes smaller value in the wavelet packet based estimation compared with that of the wavelet based estimation.

V. Conclusions

In this paper, a concrete generation method of nearly $1/f$ processes based on a realization of non-integer integral computing elements model was presented. Two effective estimation methods for nearly $1/f$ processes using a wavelet with complementary sampling and a wavelet packet with EM algorithm were proposed. In the simulations, it is shown that based on the wavelet with complementary sampling, parameters estimation can be improved even when the observed signal does not have sufficient data length. Further, it is shown that parameters estimation can be improved in a wide range of nearly $1/f$ processes based on the wavelet packet based estimation.

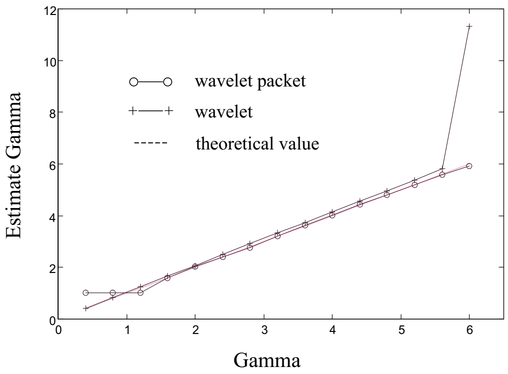


Figure 3: Comparison of estimated γ values using wavelet packet and wavelet transform. (Step:11, Data length:$2^{17} = 131072$).

References

[1] G.W. Wornell and A.V. Oppenheim: "Estimation of Fractal Signals from Noisy Measurements Using Wavelets", *IEEE Trans. Signal Processing*, vol.40, pp.611–623, 1992.

[2] G.W. Wornell: " A Karhunen-Loeve-Like Expansion for $1/f$ Processes via Wavelets", *IEEE Trans. Inform. Theory*, vol.36, pp.859–861, 1990.

[3] P. Frandrin: "On the Spectrum of Fractional Brownian Motions", *IEEE Trans. Inform. Theory*, vol.35, pp.197–199, 1989.

[4] B.B. Mandelbrot: *The Fractal Geometry of Nature*, San Francisco, CA; Freeman, 1982.

[5] K. Hashimoto, S. Takahashi and H. Amano: "Realization of Non-Integer Integral Computing Elements", *J.I.E.E.J.*, vol.89–11, no.974, pp.207–214, 1969.

[6] N.M. Laird, and A.P. Dempster and D.B. Rubin: "Maximum Likelihood from Incomplete Data via The EM Algorithm", *Ann. Roy. Stat. Soc.*, pp.1-38, Dec. 1977.

[7] R.R. Coifman and M.V. Wickerhauser: "Entropy-Based Algorithms for Best Basis Selection", *IEEE Trans. Inform. Theory*, vol.38, no.2, pp.713–718, 1992.