

TWO APPROACHES TO NON-STATIONARY LINEAR INVARIANT SYSTEMS GOVERNED BY DIFFERENTIAL EQUATIONS

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ABSTRACT

We obtain the governing differential equation for the Wigner distribution for a linear invariant system that is governed by a differential equation or the output. We use the impulse response method to prove that the solution to our equation is unique and is the Wigner distribution of the solution. Using this method we apply it to the gliding tone problem which is the response of a resonant circuit when the driving force is a quadratic modulated phase. This problem has been considered by many investigators for over fifty years and no exact solution has previously been obtained. We have obtained the *exact* Wigner distribution.

1. INTRODUCTION

We have recently developed a new approach to study linear invariant systems that are governed by an ordinary differential equation. Using this approach we have solved a hitherto unsolved problem, the gliding tone problem. This is described in section 4. The basic idea of our approach is the following. Many physical systems are represented by a linear differential equation of the following type:

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} \cdots + a_1 \frac{dx}{dt} + a_0 x = f(t) \quad (1)$$

where $x(t)$ is the state variable, which represents the state of the system, and $f(t)$ is an arbitrary forcing function. A classical interpretation of this equation, well known in signal analysis, is the system approach. With that philosophy, the differential equation is seen as a system that for a given input signal $f(t)$ returns an output signal $x(t)$. Hence, the system is interpreted as a transformation between the input and the output. Many considerations can be made directly on Eq. (1), that is, in the time domain. Other domains are

often introduced to better understand the system, particularly the Fourier and Laplace domains. However, when one is interested in studying how the instantaneous spectrum of $f(t)$ is modified by the system, a time-frequency distribution of the output is computed, such as the Wigner Distribution [3]

$$W_{x,x}(t, \omega) = \frac{1}{2\pi} \int x^*(t - \frac{1}{2}\tau) x(t + \frac{1}{2}\tau) e^{-j\tau\omega} d\tau \quad (2)$$

In our method we bypass obtaining $x(t)$ and obtain the Wigner distribution directly [4, 5] and this will be described in the next section. There are many advantages for this approach from both the practical and theoretical point of view and these will be discussed in Sect. 5. However we mention here that using this approach we have solved a hitherto unsolved problem, the gliding tone problem [6].

The possibility of studying linear time-invariant systems using the Wigner distributions was first discussed by Claasen and Mecklenbrauker [2] who derived the general relation between the Wigner distribution of the input, output and impulse response function. This method was developed by Pei and Wang [9] and also applied to time-varying system. These studies considered the general issues but did not consider the question of obtaining differential equations for the Wigner distribution.

In this paper we describe and compare the two methods and we discuss a certain fundamental issue that we call the representability problem.

It is important to notice that, although using the impulse response method it is possible to draw some interesting conclusions on the system, there are on the other hand some limitations. First, all the knowledge on how the instantaneous spectrum is modified is hidden in the Wigner of the impulse response, that acts as a "black box" between the input and the output. Then the method can be applied directly only to linear ordinary equations with constant coefficients, while important nonstationary behaviors are generated by time-varying coefficients.

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In this paper we prove that the Wigner computed with our method is the same than the one obtained with the impulse response approach. This is a crucial consideration, because it guarantees that writing a differential equation for the Wigner distribution is possible, *and* that the solution to such equation is a representable Wigner distribution.

2. THE TWO METHODS

In the following we will use Fourier transform pairs defined by ¹

$$X(\omega) = \frac{1}{\sqrt{2\pi}} \int x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{\sqrt{2\pi}} \int X(\omega) e^{j\omega t} d\omega$$

We define the convolution of two functions as

$$f(t) * g(t) = \int f(t-t')g(t')dt' \quad (3)$$

Differentiation of functions with respect to time will be indicated with this standard notation,

$$g^{(n)} = \frac{d^n}{dt^n}g(t)$$

Also, it will be convenient to define the differential operator, D

$$D = \frac{d}{dt}$$

2.1. The Wigner distribution for linear differential equations

We rewrite (1) in polynomial operator notation

$$P_n(D)x(t) = f(t) \quad (4)$$

We have proved [4] that the Wigner distribution satisfies the following partial differential equation,

$$P_n^*(A)P_n(B)W_{x,x}(t, \omega) = W_{f,f}(t, \omega) \quad (5)$$

where A and B are the following operators

$$A = \frac{1}{2} \frac{\partial}{\partial t} - j\omega \quad (6)$$

$$B = \frac{1}{2} \frac{\partial}{\partial t} + j\omega \quad (7)$$

¹All integrals go from $-\infty$ to ∞ unless otherwise indicated.

2.2. The Wigner of the impulse response

In the impulse response methods [8] one writes the solution to (1) as

$$x(t) = h(t) * f(t) \quad (8)$$

where $h(t)$ is the impulse response, that is the solution to

$$P_n(D)h(t) = \delta(t) \quad (9)$$

This corresponds in finding the particular integral of (1) when the initial conditions are all zero

$$x(-\infty) = x^{(1)}(-\infty) = \dots = x^{(n-1)}(-\infty) = 0 \quad (10)$$

Now one can approach the calculation of the Wigner distribution by taking the Wigner distribution of both sides of (8) and expressing the right hand side of the individual Wigner distribution. This has been previously done by [2] with the result that

$$W_{x,x}(t, \omega) = 2\pi W_{h,h} * W_{f,f} \quad (11)$$

$$= 2\pi \int W_{h,h}(t-t', \omega) W_{f,f}(t', \omega) dt' \quad (12)$$

The 2π is a consequence of how we define Fourier transform pairs in Sect. 2.

3. EQUIVALENCE OF THE TWO METHODS

The Representability Problem

Although we have proved that the Wigner distribution satisfies Eq. (5) there is the issue that possibly not every solution of Eq. (5) is indeed a proper Wigner distribution. We will now prove that indeed every solution of Eq. (5) is a proper Wigner distribution.

For convenience we rewrite here the main equation (4),

$$P_n(D)x(t) = f(t) \quad (13)$$

and to make the approach consistent with the impulse response approach, we set the initial conditions to zero

$$x(-\infty) = x^{(1)}(-\infty) = \dots = x^{(n-1)}(-\infty) = 0 \quad (14)$$

Now we factorize the polynomial operator in (13)

$$[D - p_1][D - p_2] \dots [D - p_n]x(t) = f(t) \quad (15)$$

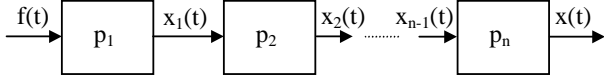


Figure 1: Block representation of equations (16). Every block is associated to an equation. For example, the equation for the first block is $x_1^{(1)}(t) + kx_1(t) = f(t)$. The initial conditions are zero for every block, and this for the first means $x_1(-\infty) = 0$.

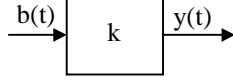


Figure 2: Basic block. The associated equation is $y^{(1)}(t) + ky(t) = b(t)$, with $y(-\infty) = 0$.

and introduce the support variables $x_1(t), \dots, x_{n-1}(t)$, (13)

$$\begin{aligned} [D - p_1]x_1(t) &= f(t) \\ [D - p_2]x_2(t) &= x_1(t) \\ &\vdots \\ [D - p_n]x(t) &= x_{n-1}(t) \end{aligned} \quad (16)$$

with the new set of initial conditions

$$x_1(-\infty) = x_2(-\infty) = \dots = x(-\infty) = 0 \quad (17)$$

This particular way of writing the problem is grafically represented in Fig. 1. Every block is associated to an equation in (16), and has the forcing function as input and the solution as output.

The basic block is represented in Fig. 2. Because the numbers p_1, p_2, \dots, p_n can be real or in general in complex conjugate pairs, the basic block is general if $k = k_R + jk_I$ is a complex number. The differential equation associated to the basic block is hence

$$\frac{dy(t)}{dt} + ky(t) = b(t) \quad (18)$$

where, as usual, the initial conditions are $y(-\infty) = 0$.

We prove that for any Wigner $W_{b,b}(t, \omega)$ the Wigner $W_{y,y}(t, \omega)$ computed with the two methods is the same. This result can be iterated from the first block to the last in Fig. 1 (that is from the first equation to the last

in (16)), to prove that for any given $W_{f,f}(t, \omega)$ in (13), $W_{x,x}(t, \omega)$ is the same for the two methods.

In particular we will obtain the following integral relation

$$W_{y,y}(t, \omega) = 2u(t)e^{-2k_R t} \frac{\sin 2t(\omega + k_I)}{\omega + k_I} \underset{t}{*} W_{b,b}(t, \omega) \quad (19)$$

3.1. Solution to the Wigner equation

Let's rewrite the differential equation (18) associated to the basic block in operator notation

$$[D + k]y(t) = b(t) \quad (20)$$

The associated Wigner equation is [4]

$$[A + k^*][B + k]y(t) = b(t) \quad (21)$$

This equation can be rewritten in the equivalent system formulation

$$\begin{aligned} [A + k^*]W_1(t, \omega) &= W_{b,b}(t, \omega) \\ [B + k]W_{y,y}(t, \omega) &= W_1(t, \omega) \end{aligned}$$

where the support function $W_1(t, \omega)$ has been introduced. We set the initial conditions to zero, that is $W_1(-\infty, \omega) = W_{y,y}(-\infty, \omega) = 0$ and we solve the two linear first order equations in cascade.

3.1.1. First equation

The first equation is

$$AW_1 + k^*W_1 = W_{b,b} \quad (22)$$

Substituting A from (6)

$$\frac{\partial W_1}{\partial t} + 2z^*W_1 = 2W_{b,b} \quad (23)$$

where

$$z^* = (k + j\omega)^* \quad (24)$$

Treating the equation as an ordinary equation with ω as a parameter, the solution is

$$W_1(t, \omega) = 2e^{-2z^*t} \int_{-\infty}^t e^{2z^*t'} W_{b,b}(t', \omega) dt' \quad (25)$$

3.1.2. Second equation

The second equation is

$$BW_{y,y} + kW_{y,y} = W_1 \quad (26)$$

Substituting B from (7)

$$\frac{\partial W_{y,y}}{\partial t} + 2zW_{y,y} = 2W_1 \quad (27)$$

Treating the equation as an ordinary equation with ω as a parameter, the solution is

$$\begin{aligned} W_{y,y}(t, \omega) &= 2e^{-2zt} \int_{-\infty}^t e^{2zt'} W_1(t', \omega) dt' \\ &= 2e^{-2zt} \int_{-\infty}^t e^{2zt'} \\ &\quad \times \left(2e^{-2z^*t'} \int_{-\infty}^{t'} e^{2z^*t''} W_{b,b}(t'', \omega) dt'' \right) dt' \\ &= 4e^{-2zt} \int u(t-t') u(t'-t'') \\ &\quad \times e^{2(z-z^*)t'} e^{2z^*t''} W_{b,b}(t'', \omega) dt' dt'' \end{aligned}$$

Let's study the integral

$$\begin{aligned} I &= \int u(t-t') u(t'-t'') e^{2(z-z^*)t'} dt' \\ &= \begin{cases} 0, & t < t'' \\ \int_{t''}^t e^{2(z-z^*)t'} dt' = \frac{e^{2(z-z^*)t} - e^{2(z-z^*)t''}}{2(z-z^*)}, & t \geq t'' \end{cases} \\ &= u(t-t'') \frac{e^{2(z-z^*)t} - e^{2(z-z^*)t''}}{2(z-z^*)} \end{aligned}$$

Substituting in $W_{y,y}$

$$\begin{aligned} W_{y,y} &= 4e^{-2zt} \int u(t-t'') \frac{e^{2(z-z^*)t} - e^{2(z-z^*)t''}}{2(z-z^*)} e^{2z^*t''} \\ &\quad \times W_{b,b}(t'', \omega) dt'' \\ &= 4 \int u(t-t'') \frac{e^{-2z^*(t-t'')} - e^{-2z(t-t'')}}{2(z-z^*)} \\ &\quad \times W_{b,b}(t'', \omega) dt'' \\ &= 2 \int u(t') \frac{e^{-2z^*t'} - e^{-2zt'}}{z-z^*} W_{b,b}(t-t', \omega) dt' \\ &= 2 \int u(t') e^{-2k_R t'} \frac{e^{j2(w+k_I)t'} - e^{-j2(w+k_I)t'}}{2j(w+k_I)} \\ &\quad \times W_{b,b}(t-t', \omega) dt' \\ &= 2 \int u(t') e^{-2k_R t'} \frac{\sin 2t'(w+k_I)}{w+k_I} W_{b,b}(t-t', \omega) dt' \\ &= 2u(t) e^{-2k_R t} \frac{\sin 2t(w+k_I)}{w+k_I} * W_{b,b} \end{aligned}$$

that is the expected integral relation (19) between the input Wigner $W_{b,b}$ and the output Wigner $W_{y,y}$.

3.2. Wigner of the impulse response

We write the solution to (18) by mean of the impulse response

$$y(t) = h(t) * b(t) \quad (28)$$

where $h(t)$ is the impulse response of the system, that is the solution of the following problem

$$\dot{h} + kh = \delta(t) \quad (29)$$

Some simple algebra shows that

$$h(t) = u(t) e^{-kt} \quad (30)$$

Taking the Wigner of both sides, we have [2]

$$W_{y,y} = 2\pi W_{h,h} * W_{b,b} \quad (31)$$

We compute the Wigner of the impulse response

$$\begin{aligned}
W_{h,h} &= \frac{1}{2\pi} \int u(t - \tau/2)u(t + \tau/2) \\
&\quad \times e^{-k^*(t-\tau/2)}e^{-k(t+\tau/2)}e^{-j\tau\omega} d\tau \\
&= \frac{1}{2\pi} e^{-2k_R t} \int u(t - \tau/2)u(t + \tau/2) \\
&\quad \times e^{-(k-k^*)\tau/2}e^{-j\tau\omega} d\tau \\
&= \frac{1}{2\pi} u(t)e^{-2k_R t} \int_{-2t}^{2t} e^{-j(w+k_I)\tau} d\tau \\
&= \frac{1}{\pi} u(t)e^{-2k_R t} \frac{\sin 2t(w + k_I)}{w + k_I}
\end{aligned}$$

Substituting in (31) we have

$$W_{y,y}(t, \omega) = 2u(t)e^{-2k_R t} \frac{\sin 2t(\omega + k_I)}{\omega + k_I} *_t W_{b,b}(t, \omega) \quad (32)$$

That is again relation (19).

4. EXAMPLE

The response of a resonant circuit when the driving force is a chirp is the “gliding tone” problem. Specifically one seeks the solution to

$$\frac{d^2 x(t)}{dt^2} + 2\mu \frac{dx(t)}{dt} + \omega_0^2 x = e^{j\beta t^2/2} \quad (33)$$

This problem is important for many reasons and it was first considered by 1948 Barber and Ursell [1] and Hok [7] and other investigators subsequent to them. No exact solution is known but the problem has been investigated by approximation methods. Using our method we have been able to obtain the exact Wigner distribution for this problem. The differential equation for the Wigner distribution is

$$[A^2 + 2\mu A + \omega_0^2][B^2 + 2\mu B + \omega_0^2]W_{x,x}(t, \omega) = W_{f,f}(t, \omega) \quad (34)$$

where

$$W_{f,f} = \delta(\omega - \beta t) = \frac{1}{|\beta|} \delta(t - \omega/\beta)$$

We have been able so solve this equation exactly.

$$\begin{aligned}
W(t, \omega) &= \frac{2}{|\beta|} \frac{u(\tau)}{z_2 - z_1} \\
&\left[\frac{1}{\bar{z}_1 - z_1} \left(\frac{e^{-2z_1\tau} - e^{-2\bar{z}_2\tau}}{\bar{z}_2 - z_1} - \frac{e^{-2\bar{z}_1\tau} - e^{-2\bar{z}_2\tau}}{\bar{z}_2 - \bar{z}_1} \right) \right]
\end{aligned}$$

$$- \frac{1}{\bar{z}_1 - z_2} \left(\frac{e^{-2z_2\tau} - e^{-2\bar{z}_2\tau}}{\bar{z}_2 - z_2} - \frac{e^{-2\bar{z}_1\tau} - e^{-2\bar{z}_2\tau}}{\bar{z}_2 - \bar{z}_1} \right) \quad (35)$$

with

$$\tau = t - \omega/\beta \quad (36)$$

and

$$\begin{aligned}
z_1 &= -j\omega + \mu - \sqrt{\mu^2 - \omega_0^2} \\
\bar{z}_1 &= j\omega + \mu - \sqrt{\mu^2 - \omega_0^2} \\
z_2 &= -j\omega + \mu + \sqrt{\mu^2 - \omega_0^2} \\
\bar{z}_2 &= j\omega + \mu + \sqrt{\mu^2 - \omega_0^2}
\end{aligned} \quad (37)$$

and where $u(t)$ is defined by

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (38)$$

Having the exact solution has allowed us to understand the gliding tone problem in an effective and direct manner. We have considered and specialized to the underdamped, overdamped, and critically damped cases. This specific details of the derivation and the discussion of the solution will be published elsewhere [6].

5. CONCLUSION

We have developed a method to obtain the governing differential equation for the Wigner distribution for a linear invariant system whose output is governed by a linear differential equation. The Wigner distribution is a real function of two variables, but not all the real functions of two variables are Wigner distributions. This is called the “representability problem”. In this paper we have compared the Wigner obtained with our method to the Wigner obtained with the impulse response method, and we have proved that the former is a representable Wigner distribution.

Using this method we have solved the “gliding tone” problem, that is the problem of finding the output of an harmonic oscillator whose input is a linear chirp. In the time domain no exact solution exists, but we have instead found the *exact* Wigner distribution.

REFERENCES

- [1] N. F. Barber and F. Ursell, “The response of a resonant system to a “gliding tone,” *Phil. Mag.*, vol. 39, pp. 345–361, 1948.

- [2] T.A.C.M Claasen and W.F.G. Mecklenbrauker, "The Wigner distribution-A tool for time-frequency signal analysis. Part I: Continuous-time signals", *Philips Journal of Research*, vol. 35, No. 3, pp. 217-250, 1980.
- [3] L. Cohen, *Time-Frequency Analysis*, Prentice-Hall, 1995.
- [4] L. Galleani, L. Cohen, "Dynamics Using the Wigner Distribution", *to appear in ICPR 2000*, September 3-8 2000, Barcelona, Spain.
- [5] L. Galleani, L. Cohen, "The Wigner Distribution for Ordinary Linear Differential Equations and Wave Equations", *to appear in IEEE SSAP 2000*, August 14-16 2000, Pocono Manor, Pennsylvania, USA
- [6] L. Galleani, L. Cohen, "On the Exact Solution to the "Gliding Tone " Problem", *to appear in IEEE SSAP 2000*, August 14-16 2000, Pocono Manor, Pennsylvania, USA
- [7] G. Hok, "Response of linear Resonant systems to Excitation of a frequency Varying Linearly with Time" *Journal of Applied Physics*, vol. 19, pp. 242-250, 1948.
- [8] A. Papoulis, *Signal Analysis*, McGraw-Hill, 1984.
- [9] S. Pei, T. Wang, "The Wigner Distribution of Linear Time-Variant Systems", *IEEE Transactions on Acoustic Speech and Signal Processing*, Vol. 36, No. 10, October 1988.