ON SOME CLASSES OF LINEAR TIME-VARYING PARAMETRIC FILTERS

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ABSTRACT

In this paper we investigate linear time-varying filters characterized by temporal- and spectral-domain parameters. Existence and uniqueness properties are presented for four classes of parametric time-varying filters: (1) the rational class; (2) the rational adjoint class; (3) the well-known ARMA class; and (4) the ARMA adjoint class. In previous work on nonstationary processes [1], we presented membership conditions on the Green's function for each of these classes. These conditions were used to determine when minimum-order parameterizations are unique and to give precise conditions under which a unique minimum-order filter is a member of one or more of these classes. In this paper, we present these results in a system theoretic framework.

1. INTRODUCTION

Linear time-invariant system theory is a mature field with wide application in signal processing, controls, and communication. However, real-world problems usually involve time-varying systems. For example, time-varying matched filters are used in the detection of signals propagating in dispersive media. Adaptive equalizers are used in wireless communication to cancel channel characteristics that vary with time. Results for time-varying systems are limited, making it difficult to treat many real-world problems.

This paper concerns four classes of linear time-varying parametric filters: (1) rational; (2) rational adjoint; (3) ARMA; and (4) ARMA adjoint. The first two classes are parameterized in the frequency domain, and the last two classes are parameterized in the time-domain. In each case, the filter is characterized by a time-varying Green's function. The rational class is closely related to the Evolutionary Spectrum (ES) presented by Priestley in [2]. It is based on an innovations system interpretation in which the ES is defined by Zadeh's frequency response function [3]. The rational adjoint class is related in the same way to the Transitory Evolutionary Spectrum (TES) discussed in [4]. In the innovations system interpretation, the TES is defined by the frequency response of the adjoint signal model studied by Kailath [5]. The well-known ARMA class has received a great deal of attention in the literature. Several researchers have investigated techniques for generating ARMA parameterizations for time-varying systems [6]. In the time-invariant case, these four classes "coalesce," making it easy to transform between frequency-domain and time-domain parameterizations. As we argued above, this is more difficult in the time-varying case. Often a transformation does not exist.

In [1] we considered nonstationary processes and investigated the fundamental properties of existence and uniqueness for each of these four classes and their intersections. Necessary conditions on the Green's function were given for membership in each of the four classes. These conditions were used to determine when minimum-order parameterizations are unique. Results on uniqueness were used to give precise conditions under which a unique minimum-order system is a member of one or more of these classes. This paper unified and extended the results of Grenier, Huang and Aggarwal, and Sills and Kamen. Grenier previously presented necessary and sufficient conditions on the Green's function for the existence of ARMA realizations in [6]. Huang and Aggarwal investigated the simultaneous existence of rational and ARMA models in [7]. Sills and Kamen investigated the existence of unique minimum-order realizations in [8].

In this paper we present results in a system theoretic framework. Figure 5 at the end of the paper summarizes the results on class membership. Although quite restrictive, these conditions do not apply to non-unique minimum-order systems; that is, the class of filters that are both rational and ARMA is expanded when restrictions on uniqueness are relaxed. This subtle, but important result means that transformations to higher-order parameterizations are not only expected to improve filter performance, but are in some cases exact representations.

The structure of the paper is as follows. After the introduction in Section 1, the paper begins in Section 2 with a review of some fundamental results from time-varying system theory. Next, the four classes of time-varying parametric systems are presented. In Section 3 we present the definition of unique minimum-order parameterizations. In Section 4 we use the results on uniqueness to find membership conditions for each of these parametric classes and their intersections. Conclusions are presented in Section 5. Throughout the paper results are presented without proof. Complete derivations are self evident from those in [1].

2. PARAMETRIC SYSTEMS

Consider the general class of linear time-varying systems as shown in Figure 1. More precisely, we consider the output of the time-varying system $g(n,m)$ given by the convolutional summation.

$$y(n) = \sum_{m=-\infty}^{\infty} g(n,m)x(m) \tag{1}$$

where $x(n)$ is an arbitrary input to the system, and the Green's functions $g(n,m)$ is the real-valued function of integer variables equal to the linear system's impulse response function defined as the system response at index n to an impulse applied at index m [9]. This function completely characterizes a time-varying system in the time domain. In this work we require that the signal model be causal, and thus $g(n,m) = 0$ for all $m > n$.

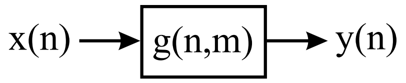

Figure 1: Block diagram of signal model.

If $g(n,m) = g(n-m,0)$ for all n and m, then as is well known, the system is time invariant and can be completely characterized by its response to a single impulse.

For this paper we require that $x(n)$ has a well-defined Fourier Transform and that there exists a function $c(n)$, with $c(n) < \infty$, such that

$$\sum_{m=-\infty}^{n} |g(n,m)| < c(n)$$

and

$$\sum_{m=n}^{\infty} |g(m,n)| < c(n)$$

Before proceding we introduce Zadeh's frequency response function defined by

$$G(e^{j\omega}, n) = \sum_{i=0}^{\infty} g(n, n-i)e^{-j\omega i} \tag{2}$$

and the adjoint frequency response function defined by

$$G(n, e^{j\omega}) = \sum_{i=0}^{\infty} g(n+i, n)e^{-j\omega i} \tag{3}$$

Note carefully the difference between the definitions of $G(e^{j\omega}, n)$ in (2) and $G(n, e^{j\omega})$ in (3). The conjugate adjoint frequency-response function is the Fourier transform of the system's future response to an impulse applied at the index n; whereas, Zadeh's frequency-response function is the Fourier transform of the system response at the index n to impulses applied in the past. The conjugate adjoint frequency-response function is named for its relationship to the adjoint system studied by Kailath [5]. Associated with each system $g(n,m)$ is an adjoint system defined by $g_a(n,m) = g(m,n)$. It is easy to show that $G(n, e^{j\omega}) = G_a(e^{-j\omega}, n)$ where

$$G_a(e^{-j\omega}, n) = \sum_{i=-\infty}^{0} g_a(n, n-i)e^{-j\omega i}$$

The conjugate adjoint frequency-response function is simply the conjugate of the frequency-response function for the adjoint system. Like the impulse-response function and Zadeh's frequency-response function, the conjugate adjoint frequency-response function completely characterizes the system.

These frequency response functions have long been used to relate the input $x(n)$ and output $y(n)$ of a linear time-varying systems [10]. The time- and frequency-domain relationships are given by

$$y(n) = \sum_{m=-\infty}^{n} g(n,m)x(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega}, n)X(e^{j\omega})e^{j\omega n}\, d\omega$$

$$Y(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x(m)G(m, e^{j\omega})e^{-j\omega m} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma(e^{j\gamma}, e^{j\omega})X(e^{j\gamma})d\gamma$$

where the bi-frequency response function $\Gamma(e^{j\gamma}, e^{j\omega})$ is defined by

$$\Gamma(e^{j\gamma}, e^{j\omega}) = \sum_{m=-\infty}^{\infty} G(m, e^{j\omega})e^{j(\gamma-\omega)m} = \sum_{m=-\infty}^{\infty} G(e^{j\gamma}, m)e^{j(\gamma-\omega)m}$$

and $X(e^{j\omega})$ and $Y(e^{j\omega})$ are the Fourier Transforms of $x(n)$ and $y(n)$ respectively.

In a continuous-time framework, the Zadeh and conjugate adjoint frequency response functions defined by (2) and (3) are special cases of a more general class of frequency response functions known as the generalized Weyl symbol[1].

We now define the four classes of time-varying filters beginning with the well-known ARMA class.

Definition 1 *A filter with Green's function $g(n,m)$ is ARMA if there exists a fixed positive integer Q and functions $\alpha_i(n)$ $i = 1, 2, \ldots, Q$ and $\beta_i(n)$ $i = 0, 1, \ldots, Q-1$ such that $y(n)$ can be expressed in the form*

$$y(n) = \sum_{i=1}^{Q} \alpha_i(n)y(n-i) + \sum_{i=0}^{Q-1} \beta_i(n)x(n-i) \tag{4}$$

where the integer Q is the filter order and the $\alpha_i(n)$ $i = 1, 2, \ldots, Q$ and $\beta_i(n)$ $i = 0, 1, \ldots, Q-1$ are the system parameters.

Next, we define the ARMA adjoint class.

Definition 2 *A filter with Green's function $g(n,m)$ is ARMA adjoint if there exists a fixed positive integer N and functions $\varphi_i(n)$ $i = 1, 2, \ldots, N$ and $\chi_i(n)$ $i = 0, 1, \ldots, N-1$ such that $y(n)$ in can be expressed in the form*

$$y(n) = \sum_{i=0}^{N-1} \chi_i(n)v(n-i) \tag{5}$$

where

$$v(n) = \sum_{i=1}^{N} \varphi_i(n)v(n-i) + x(n) \tag{6}$$

where the integer N is the order of the signal model and the $\varphi_i(n)$ $i = 1, 2, \ldots, N$ and $\chi_i(n)$ $i = 0, 1, \ldots, N-1$ are the system parameters.

Now we define the rational class.

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[1] This continuous-time construct is expressed as

$$W_a(t, e^{j\omega}) = \int_{-\infty}^{\infty} g\left(t + \left(\frac{1}{2} - a\right)\tau, t - \left(\frac{1}{2} + a\right)\tau\right) e^{-j\omega\tau} d\tau$$

The generalized Weyl symbol $W_a(t, e^{j\omega})$ depends on the coefficient a, which governs the mix of forward and backward integration over the Green's function. For the case $a = 0$, $W_a(t, e^{j\omega})$ is known as simply the Weyl Symbol. With $a = \frac{1}{2}$, $W_a(t, e^{j\omega})$ reduces to Zadeh's frequency response function and is closely related to the Evolutionary Spectrum (ES). With $a = -\frac{1}{2}$, $W_a(t, e^{j\omega})$ reduces to the conjugate adjoint frequency response function and is closely related to the Transitory Evolutionary Spectrum (TES). A thorough treatment of these constructs is given in [4]

Definition 3 *A filter with Green's function $g(n,m)$ is rational if there exists a fixed finite positive integer P and functions $a_i(n)$ $i = 1,2,\dots,P$ and $b_i(n)$ $i = 0,1,\dots,P-1$ such that the frequency-response function $G(e^{j\omega},n)$ can be expressed in the form*

$$G(e^{j\omega},n) = \frac{\sum_{i=0}^{P-1} b_i(n)e^{-j\omega i}}{1-\sum_{i=1}^{P} a_i(n)e^{-j\omega i}} \tag{7}$$

The integer P is the order of the filter and the $a_i(n)$ $i = 1,2,\dots,P$ and $b_i(n)$ $i = 0,1,\dots,P-1$ are the filter parameters.

Finally we define the rational adjoint class.

Definition 4 *A filter with Green's function $g(n,m)$ is adjoint rational if there exists a fixed finite positive integer M and functions $\gamma_i(n)$ $i = 1,2,\dots,M$ and $\zeta_i(n)$ $i = 0,1,\dots,M-1$ such that the conjugate adjoint frequency-response function $G(n,e^{j\omega})$ of the signal model can be expressed in the form*

$$G(n,e^{j\omega}) = \frac{\sum_{i=0}^{M-1} \zeta_i(n)e^{-j\omega i}}{1-\sum_{i=1}^{M} \gamma_i(n)e^{-j\omega i}} \tag{8}$$

where the conjugate adjoint frequency-response function is defined by (3). The integer M is the order of the filter and the $\gamma_i(n)$ $i = 1,2,\dots,M$ and $\zeta_i(n)$ $i = 0,1,\dots,M-1$ are the filter parameters.

It is evident from the definitions given above that the rational class is characterized by frequency-domain parameters while the ARMA class is characterized by parameters associated with the time domain. Due to their simple recursive structure, ARMA filters are easier to implement on a computer. However, when specifying a filter's characteristics, it is often easier to work in the frequency domain.

For the time-invariant case, as is well known, it is easy to transform between frequency- and time-domain parameterizations since in this case the parameters are constant and

$$\alpha_i(n) = \varphi_i(n) = \gamma_i(n) = a_i(n) = a_i, \quad for\ all\ n,\ and\ i$$

$$\beta_i(n) = \chi_i(n) = \zeta_i(n) = b_i(n) = b_i, \quad for\ all\ n,\ and\ i$$

Unfortunately, it is difficult to transform between time- and frequency-domain parameterizations in the time-varying case. One reason is that the four parametric classes exhibit different transitory behavior. The numerical example that follows compares the different transitory characteristics that can be realized by each of the four classes in response to variations in the system parameters. In this case the parameters are varied in a "step-like" manner.

Example 5 *Consider the four time-varying systems characterized by the parameters*

$$\alpha_i(n) = \varphi_i(n) = \gamma_i(n) = a_i(n), \quad for\ all\ n,\ and\ i = 1,2$$

$$\beta_i(n) = \chi_i(n) = \zeta_i(n) = b_i(n), \quad for\ all\ n,\ and\ i = 0,1$$

where

$$a_1(n) = p(n) + p^*(n) \qquad a_2(n) = -p(n)p^*(n)$$

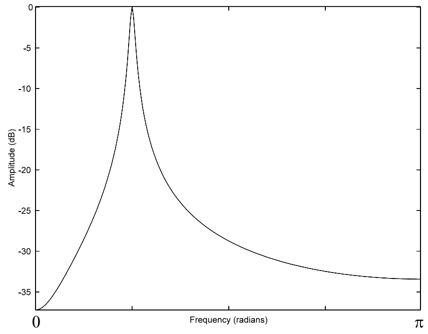
$$b_0(n) = \frac{1}{1-|p(n)|^2} \qquad b_1(n) = \frac{z(n)}{1-|p(n)|^2}$$

and

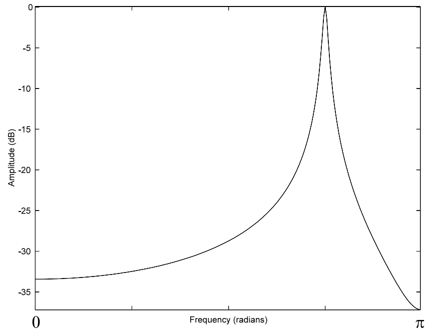
$$p(n) = \begin{cases} 0.98e^{j\frac{\pi}{4}}, & n < 0 \\ 0.98e^{j\frac{3\pi}{4}}, & n \geq 0 \end{cases}$$

$$z(n) = \begin{cases} 0.8, & n < 0 \\ -0.8, & n \geq 0 \end{cases}$$

For the given parameters, the rational frequency response $G(e^{j\omega},n)$ defined by equation (7) is shown in Figures 2(a) and 2(b). It is evident from these figures that the four parametric systems are characterized by parameters that change instantaneously from a "low-frequency" model to a "high-frequency."



(a) $n < 0$.



(b) $n \geq 0$.

Figure 2: Frequency response $G(e^{j\omega},n)$.

Figures 3(a) and 3(b) can be used to implement the rational and rational adjoint filters respectively, where we have used the step function

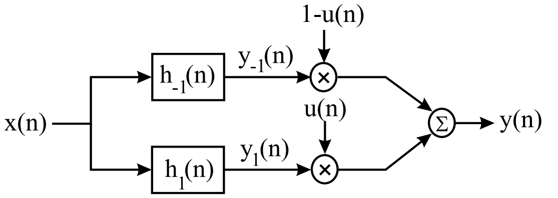
$$u(n) = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$

and the set $\{h_k(n) | k \in \mathcal{Z}\}$ is replaced with $\{h_{-1}(n), h_1(n)\}$ since

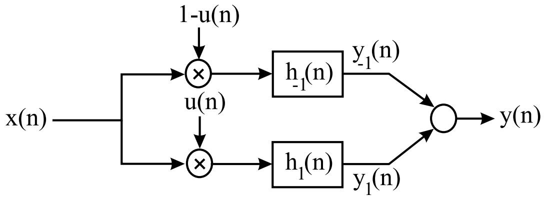
$$h_k(n) = \begin{cases} h_{-1}(n) & k < 0 \\ h_1(n) & k \geq 0 \end{cases}$$

for both the rational and rational adjoint case (where $\mathcal{Z}$ is the set of integers).

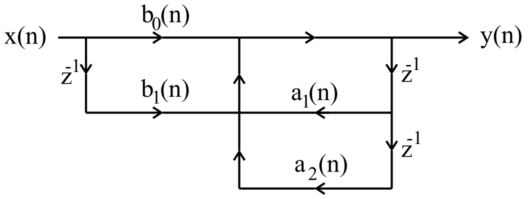
The ARMA and ARMA adjoint systems can be implemented using the recursive structures given in Figures 3(c) and 3(d) respectively.



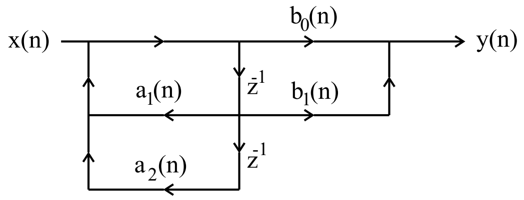
(a) Rational.



(b) Rational adjoint.

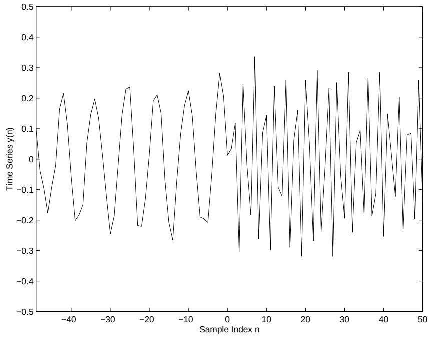


(c) ARMA.

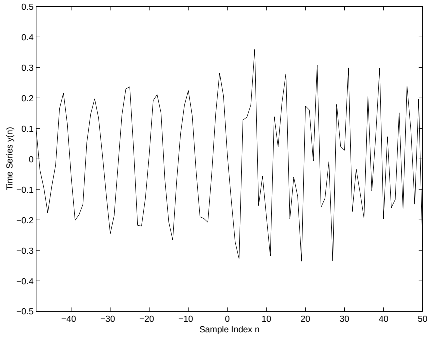


(d) ARMA adjoint.

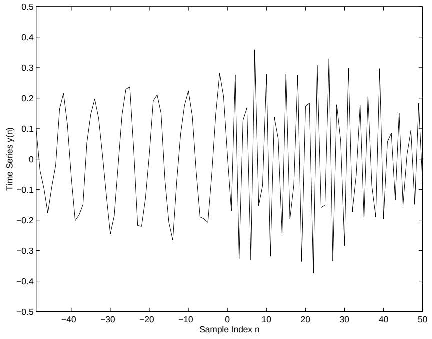
Figure 3: Filter structures.



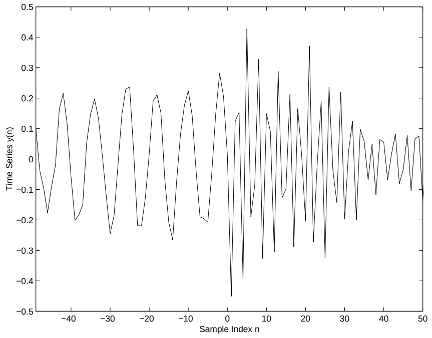
(a) Rational realization.



(b) Rational adjoint realization.



(c) ARMA realization.



(d) ARMA adjoint realization.

Figure 4: Transient responses.

Figures 4(a), 4(b), 4(c), and 4(d) show the filter response of each of the four systems to a white noise input. The input noise sequence is identical in each of the four realizations and as a result the four realizations are identical for $n < 0$. The rational, ARMA, and ARMA adjoint realizations feature similar abrupt transitions from low to high frequency at $n = 0$, except for the phase discontinuity at the transition point in the rational case. The rational adjoint realization shown in Figure 4(b) features a slow decay of the low-frequency component concurrent with a slow rise of the high-frequency component. The transition from low to high frequency does not contain intermediate frequency components. The four figures illustrate the different transient-response (step-response) characteristics of the four classes.

Note that it would be difficult to find a second-order ARMA model equivalent to the rational adjoint model, but finding an equivalent higher-order ARMA model would not a problem. Indeed, consider the parallel combination of one second-order low-frequency ARMA model with another second-order high-frequency ARMA model. In this case the poles of both second-order models are constant, and the zeros vary to switch between the two models as required. The parallel combination of these two second-order models can be realized by a single fourth-order ARMA model. Similar arguments hold for the rational and ARMA adjoint models.

3. UNIQUE MINIMUM-ORDER PARAMETERIZATIONS

In the following we consider the existence and uniqueness of parametric systems. For the sake of brevity we present results for the rational class only. The results can be extended to the rational adjoint, ARMA, and ARMA adjoint classes.

The following lemma gives a condition on the Green's function that must be satisfied if a system is rational.

Lemma 6 *A system is rational if and only if there exists an integer P and functions $a_1(n), \ldots, a_P(n)$ such that the infinite set of equations given by*

$$g(n, n-(P+i)) = \sum_{m=1}^{P} a_m(n) g(n, n-(P+i-m))\,,\ \ i = 0, 1, \ldots \tag{9}$$

are satisfied for all integers n where $g(n, m)$ is the impulse-response function of the signal model.

Rational systems do not necessarily have unique parameters. Before the question of uniqueness is addressed, the concept of minimum order is precisely defined.

Definition 7 *The smallest integer P for which there exists parameter functions $a_1(n), \ldots, a_P(n)$ such that (9) is satisfied for all integers n, is the minimum order of the system with Green's function $g(n, m)$.*

One might conjecture that a minimum-order rational parameterization is necessarily unique; however, the following example contradicts this claim and shows that there can be many different minimum-order parameterizations for a given system.

Example 8 *Given a constant value $|a| < 1$, consider the system given by the impulse-response function $g(n, n-i) = a^i$, for $i = 0, 1, 2, \ldots$, and all n but with the exception that $g(0, 0) = 2$. This time-varying system has a minimum order of two, and it can be characterized by any of an infinite number of second-order rational signal models with parameters*

$$\begin{array}{lrcl@{\qquad}lrcl}
n \neq 0: & a_1(n) & = & c_1 & n = 0: & a_1(n) & = & a \\
 & a_2(n) & = & a^2 - c_1 a & & a_2(n) & = & 0 \\
 & b_0(n) & = & 1 & & b_0(n) & = & 2 \\
 & b_1(n) & = & a - c_1 & & b_1(n) & = & -a
\end{array}$$

where c_1 is any real number. For each choice of c_1, a different parameterization is obtained.

The concept of pointwise minimum order is used next to construct a particular parameterization among the class of minimum-order parameterizations.

Definition 9 *The function $P(n)$ is the pointwise minimum order of the rational system with impulse response $g(n, m)$, if $P(n)$ is a bounded function of n and if for any given integer n, $P(n)$ is the smallest integer for which there exists parameter values $a_1(n), \ldots, a_{P(n)}(n)$ that satisfy the infinite set of equations given by*

$$g(n, n-P(n)-i) = \sum_{m=1}^{P(n)} a_m(n) g(n, n-P(n)-i+m), i = 0, 1, \ldots \tag{10}$$

A condition is now given under which the minimum-order rational parameterization is unique.

Proposition 10 *A rational system has only one parametric characterization with minimum order P if and only if the pointwise minimum-order rational parameterization has common pointwise minimum order $P(n) = P$ for all n.*

4. CLASS INTERSECTION

In this section we consider unique minimum-order rational, rational adjoint, ARMA, and ARMA adjoint parameterizations. We investigate the intersection of these classes by considering each combination of pairs in turn. We find that a system can be a member of all four classes only under very restrictive conditions.

Figure 5 presents necessary conditions for the intersection of two or more parametric systems. This figure shows the unique minimum-order rational, rational adjoint, ARMA, and ARMA adjoint classes. Rational systems are ARMA only if the denominator coefficients $a_1(n), \ldots, a_P(n)$ are constant. This condition is not sufficient as illustrated by the following example.

Example 11 *Suppose a system has frequency-response function*

$$G(e^{j\omega}, n) = \frac{b_0(n)}{1 - ae^{-j\omega n}}$$

Clearly the denominator coefficient is constant for this example. It is easily verified that the impulse-response function for the signal model is given by

$$g(n, n-i) = b_0(n) a_i$$

If $b_0(n) \neq 0$ for all n, then the system has the ARMA representation

$$\mathbf{v}(n) = a\frac{b_0(n)}{b_0(n-1)}\mathbf{v}(n-1) + b_0(n)\varepsilon(n)$$

Hence in this example where the denominator coefficient is constant, the condition for existence of an ARMA representation depends on the numerator coefficient $b_0(n)$.

Moving in a counter-clockwise direction about Figure 5 the conditions for joint membership are given. An ARMA system is rational adjoint only if the autoregressive coefficients $\alpha_1(n), \ldots, \alpha_Q(n)$ are constant. Rational adjoint systems are ARMA adjoint only if the autoregressive coefficients $\varphi_1(n), \ldots, \varphi_N(n)$ are constant. An ARMA adjoint system is rational adjoint only if the denominator coefficients $\gamma_1(n), \ldots, \gamma_M(n)$ are constant.

Finally, a system is rational and rational adjoint only if both sets of denominator coefficients, namely $a_1(n), \ldots, a_P(n)$ and $\gamma_1(n), \ldots, \gamma_M(n)$, are constant. It follows directly that any system that is both rational and rational adjoint, must in turn be both ARMA and ARMA adjoint. However, it is easy to show that there exist systems that are both ARMA and ARMA adjoint but are neither rational or rational adjoint. For example consider a time-varying first-order ARMA system with varying $\alpha_1(n)$ but constant moving average parameter $\beta_0(n) = 1$. This system is ARMA adjoint with parameter $\varphi_1(n) = \alpha_1(n)$, but it is not rational adjoint since $\alpha_1(n)$ is not constant. Further, it is not rational since $\varphi_1(n)$ is not constant.

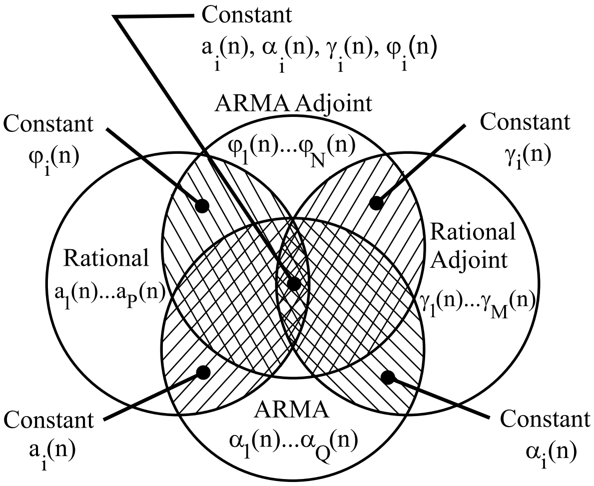


Figure 5: Sets of unique minimum-order time-varying parametric systems.

As a final example, we point out that even if a time-varying system lies within the intersection of one or more of these parametric classes, it does not imply that the systems have identical parameters.

Example 12 *Consider the time-varying parametric system with the following moving average signal model:*

$$y(n) = \beta_0(n)x(n) + \beta_1(n)x(n-1)$$

Clearly $y(n)$ is an ARMA system with constant AR coefficients. Further, a rational adjoint characterization exists since the recursive coefficients are zero. However, the parameters of the rational adjoint model are not necessarily equal to the parameters of the ARMA model. In this case the rational adjoint model is given by

$$G(n, e^{j\omega}) = \zeta_0(n) + \zeta_1(n)e^{-j\omega}$$

where

$$\zeta_0(n) = \beta_0(n) \qquad \zeta_1(n) = \beta_1(n+1)$$

5. CONCLUSIONS

In this paper we examined four classes of time-varying parametric systems: (1) rational; (2) rational adjoint; (3) ARMA; and (4) ARMA adjoint. We began with a review of the fundamental results from time-varying system theory. The concepts of minimum order and pointwise minimum order were defined and then used to present conditions under which parameterizations are unique. We then presented necessary conditions on unique parameterizations that must be satisfied for a system to be a member of two or more of these classes. Although the results in the paper show that the class of systems intersecting two or more of these sets is fairly restrictive, it is easy to demonstrate using the system in Example 8 that there is a larger class of systems that is simultaneously rational, rational adjoint, ARMA, and ARMA adjoint for which the coefficients $a_1(n), \ldots, a_P(n)$, $\alpha_1(n), \ldots, \alpha_Q(n)$, $\gamma_1(n), \ldots, \gamma_M(n)$, and $\varphi_1(n), \ldots, \varphi_N(n)$ are not constant over time. Of course these are not unique minimum-order parameterizations, but nonetheless are of interest. Furtherwork is required to determine what conditions are necessary to allow the simultaneous existence of three of the four classes, if such a scenerio is possible.

6. REFERENCES

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