SMOOTH WAVELET TIGHT FRAMES WITH ZERO MOMENTS: DESIGN AND PROPERTIES

Ivan W. Selesnick

Electrical Engineering
Polytechnic University
6 Metrotech Center
Brooklyn, NY 11201
selesi@taco.poly.edu

ABSTRACT

This paper takes up the design of wavelet tight frames that are analogous to Daubechies orthonormal wavelets — that is, the design of minimal length wavelet filters satisfying certain polynomial properties, but now in the oversampled case. The oversampled dyadic DWT considered in this paper is based on a single scaling function and two distinct wavelets. Having more wavelets than necessary gives a closer spacing between adjacent wavelets within the same scale. As a result, the transform (like Kingsbury's dual-tree DWT) is nearly shift-invariant. The oversampled DWT presented here is redundant by a factor of only 2, independent of the number of levels. In comparison, the redundancy of the undecimated DWT grows with the number of levels. Gröbner bases are used to obtain the solutions to the nonlinear design equations. It is also shown that the maximally flat FIR filters described by Herrmann arise here, as they do (in the half-band instance) in the Daubechies case.

1. INTRODUCTION

This paper introduces new wavelet tight frames based on iterated oversampled FIR filter banks. In particular, this paper takes up the design of systems that are analogous to Daubechies orthonormal wavelets [4] — that is, the design of minimal length wavelet filters satisfying certain polynomial properties, but now in the oversampled case. It should be noted that in the oversampled case, if a wavelet is very smooth it does not mean that its moments $\int t^k\, \psi(t)\, dt$ are necessarily zero (or small) for $k > 0$, in contrast to the orthonormal case. The smoothness and zero moments properties are not as closely related as they are for the critically sampled case. Like the examples by Chui and He [2] and Ron and Shen [13], the wavelets presented below are much smoother than what can be achieved in the critically sampled case, however, in this paper the zero moments properties of the wavelets are also taken into account. For a given number of wavelet moments and a given number of zeros at $z = -1$ of the scaling filter $H_0(z)$, the wavelets presented below are of minimal length.

This work was supported by the NSF under CAREER grant CCR-987452

The oversampled DWT presented in this paper differs from the undecimated DWT (UDWT). The undecimated DWT generates a truly shift-invariant discrete transform, however, the UDWT has an expansion-factor of $\log N$: it expands an N-sample data vector to $N \log N$ samples. On the other hand, the oversampled DWT presented here (like Kingsbury's dual-tree DWT [8]) expands an N-sample data vector to $2N$ samples — independent of the number of scales over which the signal decomposition is performed. While it does not yield an exactly shift-invariant discrete transform, like the dual-tree DWT, it is more nearly shift-invariant than the critically sampled DWT can be. [1]

In [2, 13] methods are given to generate a tight wavelet frame corresponding to a specified refinable function (scaling function). The approach taken in this paper is to treat the scaling and wavelet functions together as unknown. The nonlinear design equations that arise are then solved using Gröbner bases. As Gröbner bases are used in this paper to carry out the design, we are able to obtain zero wavelet moments for wavelets of minimal length, in contrast to earlier work on wavelet tight frames of this type. Although the high computational and memory cost of Gröbner bases limits their utility, we are able to obtain solutions of practical interest. In addition, software for Gröbner bases is improving with time.

Because the frames described in this paper are based on iterated FIR filter banks, a fast discrete frame transform is simple to implement. This paper considers exclusively *tight* frames.

2. PRELIMINARIES

Each of the wavelet tight frames to be developed in this paper will be based on a single scaling function $\phi(t)$ and two distinct wavelets $\psi_1(t)$ and $\psi_2(t)$. (We label the wavelets as ψ_1, ψ_2 instead of ψ_0, ψ_1 as it will simplify notation.) Following the theory of dyadic wavelet bases, the scaling

[1] Kingsbury's dual-tree DWT is designed to act as a complex wavelet transform and can thus be used to construct directional 2D filter banks. The oversampled DWT described here does not have this ability.

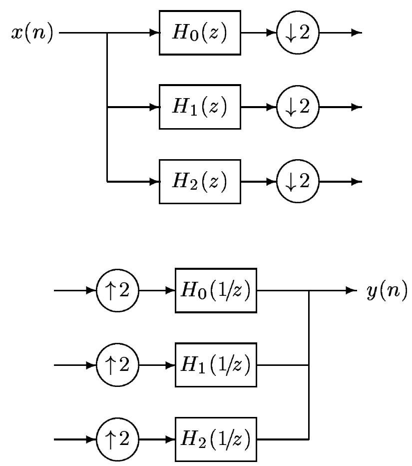

Figure 1: An oversampled analysis and synthesis filter bank.

space $\mathcal{V}_j$ and the wavelet spaces $\mathcal{W}_{i,j}$ are defined as

$$\mathcal{V}_j = \operatorname*{Span}_{n \in \mathrm{Z}}\{\phi(2^j t - n)\} \tag{1}$$

$$\mathcal{W}_{i,j} = \operatorname*{Span}_{n \in \mathrm{Z}}\{\psi_i(2^j t - n)\}, \quad i = 1, 2. \tag{2}$$

(Of course, dyadic wavelet *bases* are based on a single scaling function ϕ and a single wavelet ψ. The extra wavelet here makes this system an overcomplete one.) Following the multiresolution framework, one asks that these signal spaces be nested: $\mathcal{V}_0 \subset \mathcal{V}_1, \mathcal{W}_{1,0} \subset \mathcal{V}_1, \mathcal{W}_{2,0} \subset \mathcal{V}_1$. It follows that ϕ, ψ_1, ψ_2 satisfy the dilation and wavelet equations

$$\phi(t) = \sqrt{2} \sum_n h_0(n)\, \phi(2t - n)$$

$$\psi_i(t) = \sqrt{2} \sum_n h_i(n)\, \phi(2t - n), \quad i = 1, 2.$$

Corresponding to ϕ, ψ_1, ψ_2, we have the scaling filter $h_0(n)$, the two wavelet filters $h_1(n)$ and $h_2(n)$, and the oversampled filter bank illustrated in Figure 1. The transfer function $H_i(z)$ is given by $\sum_n h_i(n) z^{-n}$. Note that through out the paper, $t \in \mathrm{R}$, $i, j, k, l, m, n \in \mathrm{Z}$.

2.1. Frame Conditions

To design tight frames based on FIR filters h_i, it is helpful to write the tight frame conditions in terms of the coefficients h_i. By standard multirate identities, we have the perfect reconstruction conditions

$$H_0(z)\, H_0(1/z) + H_1(z)\, H_1(1/z) + H_2(z)\, H_2(1/z) = 2, \tag{3}$$

and

$$H_0(-z)\, H_0(1/z) + H_1(-z)\, H_1(1/z) + H_2(-z)\, H_2(1/z) = 0. \tag{4}$$

2.2. Zeros at $\omega = 0$, $\omega = \pi$

Let K_0 denote the number of zeros $H_0(e^{j\omega})$ has at $\omega = \pi$. For $i = 1, 2$, let K_i denote the number of zeros $H_i(e^{j\omega})$ has at $\omega = 0$.

$$(z+1)^{K_0} \mid H_0(z) \tag{5}$$

$$(z-1)^{K_1} \mid H_1(z) \tag{6}$$

$$(z-1)^{K_2} \mid H_2(z) \tag{7}$$

For orthonormal bases ($\psi_2(t) = 0$), it is necessary that $K_0 = K_1$, so no distinction need be made between K_0 and K_1. However, for tight wavelet frames of the form (2), it is not necessary that $K_0 = K_1 = K_2$. K_0 denotes the degree of polynomials representable by shifts of $\phi(t)$ and is related to the smoothness of $\phi(t)$. K_1 and K_2 denote the number of zero moments of the wavelet filters $h_1(n)$ and $h_2(n)$, provided $K_0 \geq K_1$, and $K_0 \geq K_2$.

The value of K_0 influences the degree of smoothness of ϕ (and therefore of ψ_i). On the other hand, the values K_1 and K_2 indicate what polynomials are annihilated (compressed) by the given signal expansion. In contrast to orthonormal wavelet bases, with a tight frame one has the possibility to control these parameters more freely. If it is desired for a given class of signals that the wavelets have two zero moments (for example), then the remaining degrees of freedom can be used to achieve a higher degree of smoothness by making K_0 greater than K_1 and K_2.

Although the values K_i need not all be equal, we can derive the following constraint:

$$\text{length}\, h_0 \geq K_0 + \min(K_1, K_2). \tag{8}$$

This is obtained from

$$|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 + |H_2(e^{j\omega})|^2 = 2 \tag{9}$$

which is a consequence of the tight frame condition (3). So the minimum length of h_0 is $K_0 + \min(K_1, K_2)$. In the orthonormal case $K_0 = K_1$ and $K_2 = \infty$ (as $h_2 = 0$), which gives the minimum length of h_0 to be $2K_0$, which is consistent with Daubechies' orthonormal filters.

3. NEW EXAMPLES

We seek to design FIR filters h_0, h_1, h_2 that generate tight frames of the form described in (2). We seek the shortest filters h_i having a prescribed number of zeros at $z = -1$ and $z = 1$ (specified by the values K_i) that satisfy the tight frame conditions (3,4). In the examples, we ask that $K_1 = K_2$. If they are unequal, then one wavelet annihilates more polynomials than the other, or one wavelet is doing 'more work' than the other.

Note that the conditions (3,4) are nonlinear equations in the filter coefficients $h_i(n)$. For the design problems considered below, these nonlinear design equations will be handled using Gröbner bases, a powerful but computationally expensive tool from computational algebraic geometry [3]. In a loose sense, Gröbner bases extend the Gaussian elimination of variables to polynomial systems of equations. The Gröbner bases are too big to include in the paper, but they are available on the author's webpage. For previous applications of Gröbner bases to the design of wavelets and filters, see for example [5, 10, 11, 15, 16, 17].

3.1. Example 1 — $(K_0, K_1, K_2) = (3, 1, 1)$

For the first example, we ask that $K_0 = 3$, $K_1 = K_2 = 1$. It was found that the shortest filters h_0, h_1, h_2 satisfying (3,4) are of length 4, 4, and 2, respectively. The filters are given by

$$H_0(z) = \frac{\sqrt{2}}{8} \left(1 + z^{-1}\right)^3$$

$$H_1(z) = \frac{\sqrt{2}}{8} \left(1 - z^{-1}\right) \left(1 + 4\, z^{-1} + z^{-2}\right)$$

$$H_2(z) = \frac{\sqrt{6}}{4} \left(1 - z^{-1}\right)$$

This system was found independently in [2] already. The filters and the wavelet functions are illustrated in Figure 2. $\phi(t)$ is symmetric, $\psi_i(t)$ are anti-symmetric. Notice that $\phi(t)$ is especially smooth given its support. On the other hand, it is important to note that the wavelets have only 1 zero moment: $\int \psi_i(t)\, dt = 0$, but $\int t\, \psi_i(t)\, dt \neq 0$. The analysis filter bank annihilates constant signals $x(n) = c$, but does not annihilate ramp signals $x(n) = c \cdot n$.

Due to the simple form of h_0, the scaling function $\phi(t)$ is a B-spline, obtained by convolving a square pulse $p(t)$ with itself: $\phi(t) = p(t) * p(t) * p(t)$. Therefore, closed form expressions for ϕ, ψ_i can be obtained for this example.

Also, notice that the length of h_0 is equal to $K_0 + \min(K_1, K_2)$, which by (8) is the minimal length possible.

3.2. Example 2 — $(K_0, K_1, K_2) = (5, 2, 2)$

For the second example, we ask that $K_0 = 5$, $K_1 = K_2 = 2$. It was found that the shortest filters h_0, h_1, h_2 satisfying (3,4) are of length 7, 7, and 5, respectively.

This example does not appear to admit simple expressions for the coefficients $h_i(n)$, as the nonlinear design equations are much more complex. However, by utilizing Gröbner basis methods [3] it is possible to obtain exact expressions for $h_i(n)$. (*Singular* [6] was used for the Gröbner basis calculations.) The original design equations have only rational coefficients, and we were able to obtain *explicit* expressions for $h_i(n)$ in terms of radicals. The expressions obtained for $h_i(n)$ are too long to include here, but are available from the author.

As in the orthonormal case, there are multiple solutions to this problem. However, in contrast to the orthonormal case, (i) the distinct solutions do not all share the same autocorrelation, and (ii) not all of the spectral factors of each autocorrelation are solutions.

In this example, there are 4 distinct solutions, not counting their time-reversals ($h_i(-n)$) and negations ($-h_i(n)$)). One of those 4 solutions is shown in Figure 3. The other 3 solutions are tabulated on the author's webpage. None of the 4 solutions are symmetric. None of the solutions are splines as in the first example.

3.3. Example 3 — $(K_0, K_1, K_2) = (7, 3, 3)$

For the third example, we ask that $K_0 = 7$, $K_1 = K_2 = 3$, for which the shortest filters h_0, h_1, h_2 forming a tight frame are of length 10, 10, and 8, respectively.

Again, we used Gröbner bases to solve the nonlinear design equations and we obtained explicit solutions for h_i in terms of radicals. There are exactly 8 distinct solutions, not counting their time-reversals ($h_i(-n)$) and negations ($-h_i(n)$)). One of those 8 solutions is shown in Figure 4. The other 7 solutions are tabulated on the author's webpage.

4. MAXIMALLY FLAT FILTERS

Due to the constraint (9), if $H_1(z)$ and $H_2(z)$ have at least M zeros at $z = 1$, then the first $M - 1$ derivatives of $|H_0(e^{j\omega})|$ at $\omega = 0$ must be zero. Therefore, the minimal length lowpass filter h_0 can be obtained by spectral factorization of a maximally flat symmetric FIR filter, a family of filters originally described by Herrmann [7]. Specifically, letting $M = \min(K_1, K_2)$, one has

$$|H_0(e^{j\omega})|^2 = 2\,(1 - y)^{K_0} \sum_{k=0}^{M-1} \binom{K_0 - 1 - k}{k} y^k \qquad (10)$$

where $y = \frac{1}{2}(1 - \cos\omega)$. When $M = K_0$, this formula specializes to the Daubechies polynomial; that is the polynomial that is used in Daubechies' construction of short orthonormal wavelets [4]. The Daubechies polynomial is the halfband instance of the maximally flat filter. The filter in (10) is of the same maximally flat family, but rather than being halfband, it has instead a higher degree of flatness at $\omega = \pi$ than it does at $\omega = 0$. That makes the passband more narrow than the stopband and increases the smoothness of $\phi(t)$.

While (10) yields directly a formula for $|H_0(e^{j\omega})|^2$ from which $h_0(n)$ can be obtained through spectral factorization, it does not yield the filters $h_1(n)$, $h_2(n)$.

5. NEAR SHIFT-INVARIANCE

For the type of tight frame presented above, the (idealized) time-frequency localization of the wavelets are indicated in Figure 5. Having more wavelets than necessary gives a closer spacing between adjacent wavelets within the same scale. It is this property that makes this system less shift-sensitive than a nonredundant system. Each scale in Figure 5 is represented by twice as many wavelets as in the critically sampled case. In this way, the tight frame DWT approximates the continuous wavelet transform more closely than does the critically sampled DWT, and consequently it is more robust to shifts than the critically sampled DWT.

Kingsbury demonstrated the near shift-invariance of the dual-tree DWT in [8, 9] by reconstructing a shifted discrete-time step function $u(n - n_0)$ from only its wavelet coefficients at a single scale j. Varying the shift n_0 in increments of 1, the results reveal the shift-varying properties of the system. Following the same procedure, for $j = 1, 2, 3, 4$, the left hand side of Figure 6 illustrates the near shift-invariance of the oversampled filter bank of Example 3. For comparison, the right hand side uses Daubechies' orthonormal basis D_5 (length $h_0 = 10$) [4]. The top panels show the reconstruction from only the scaling coefficients at level $j = 4$. Although the overcomplete expansion of Example 3 is not as shift-insensitive as the dual-tree DWT presented in [9],

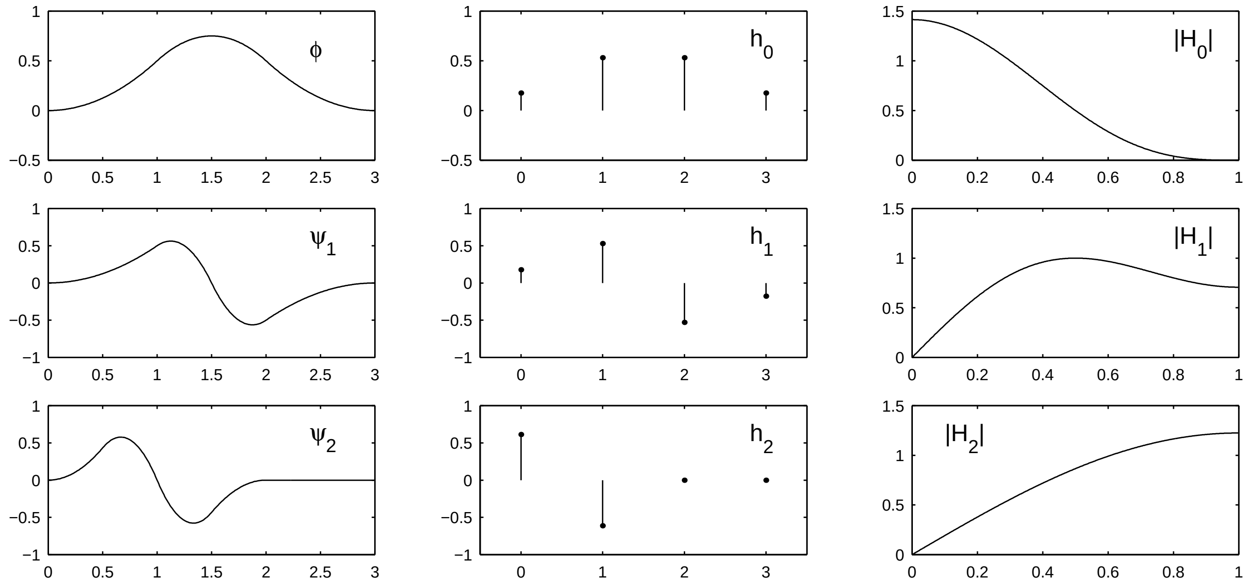

Figure 2: The generators of a wavelet tight frame with parameters $K_0 = 3$, $K_1 = K_2 = 1$.

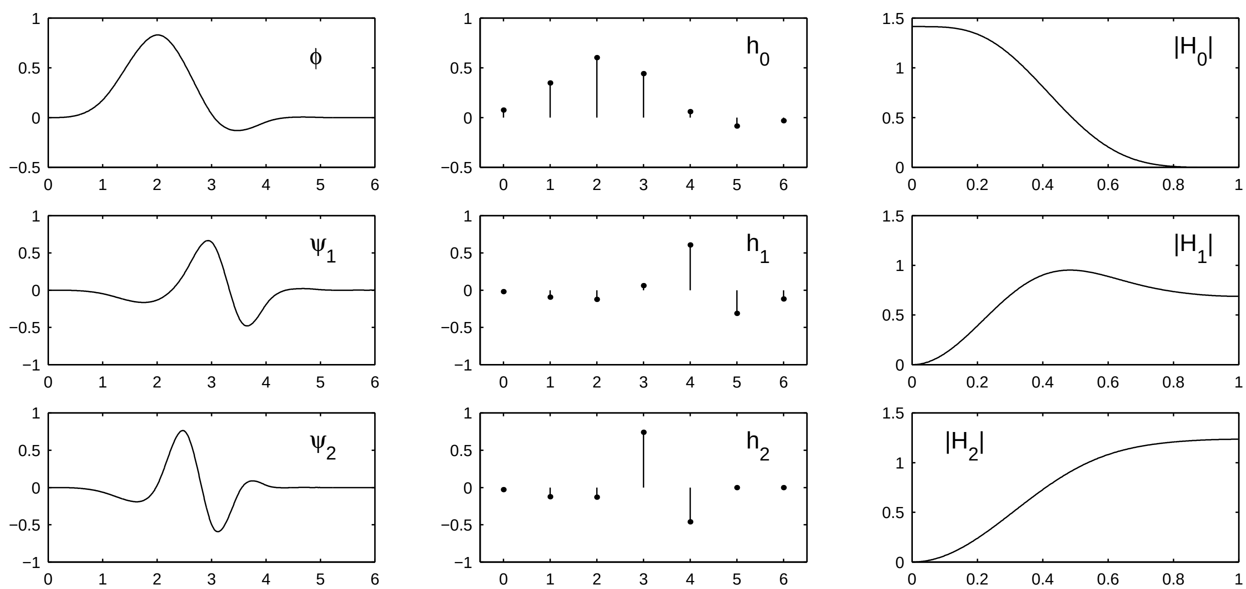

Figure 3: The generators of a wavelet tight frame with parameters $K_0 = 5$, $K_1 = K_2 = 2$.

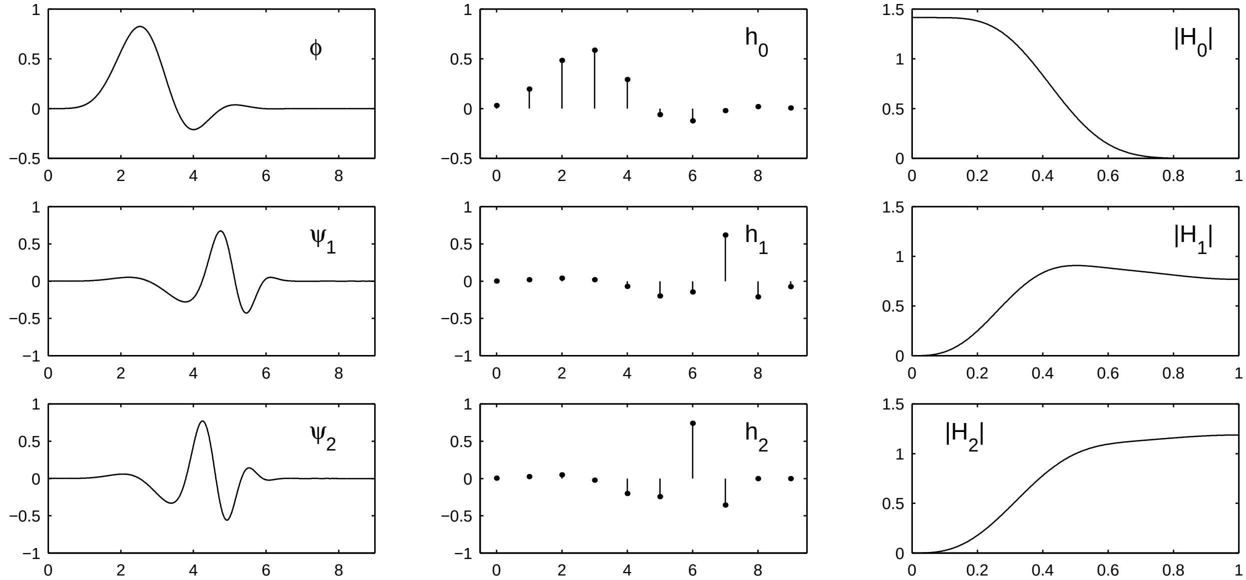

Figure 4: The generators of a wavelet tight frame with parameters $K_0 = 7$, $K_1 = K_2 = 3$.

it is much less shift-sensitive than the orthonormal basis, as illustrated in Figure 6.

It should be noted that other orthonormal bases may be less shift-sensitive than Daubechies' bases, for example those designed in [1]; however, the shift-sensitivity properties of orthonormal wavelet bases are naturally limited in comparison with tight wavelet frames.

6. CONCLUSION

Kingsbury showed that the shift-sensitivity of the DWT can be dramatically improved by using a dual-tree, an overcomplete expansion that is redundant by a factor of 2 only. So motivated, this paper considered the design of wavelet tight frames based on iterated oversampled filter banks as in [2, 13, 14]. In particular, we consider the design of wavelet tight frames that are analogous to Daubechies orthonormal wavelets bases. As the number of zeros $H_0(z)$ has at $\omega = \pi$ need not equal the number of zeros $H_1(z)$ and $H_2(z)$ have at $\omega = 0$, a greater design freedom is available, than in the orthonormal case. Although the resulting design equations are nonlinear, Gröbner bases can be used to obtain the solutions. By asking that $K_0 > K_1, K_2$, wavelets are obtained that are very smooth in comparison with orthonormal wavelet bases. Like the dual-tree DWT of Kingsbury, the overcomplete DWT described above is less shift-sensitive than an orthonormal wavelet basis (and in the 2D case has fewer rectangular artifacts).

The complete solutions to the examples examined in the paper and other examples are available on the author's webpage: `taco.poly.edu/selesi`. Also available are the *Singular* programs for obtaining the Gröbner bases from which the solutions are obtained.

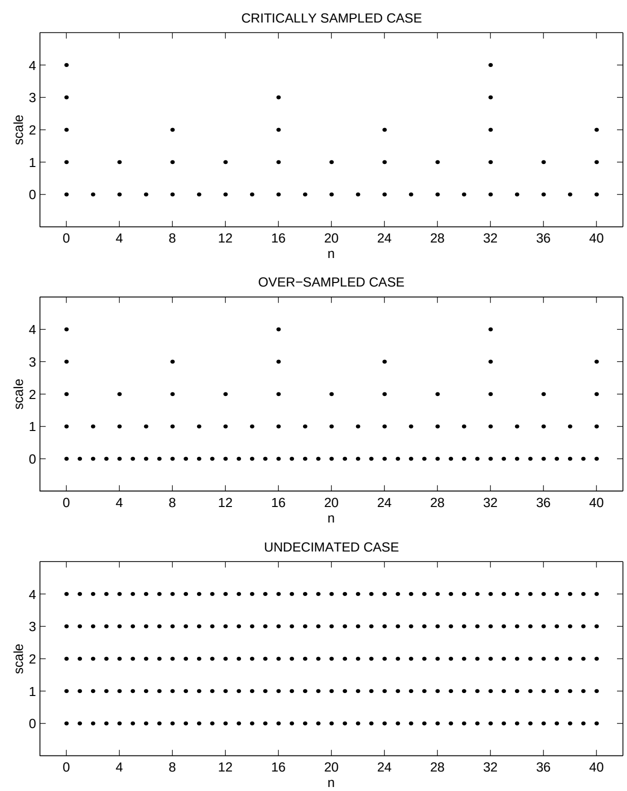

Figure 5: Idealized time-frequency localization diagrams. The tight frame gives a denser sampling of the time-frequency plane. But unlike the UDWT, maintains the same style.

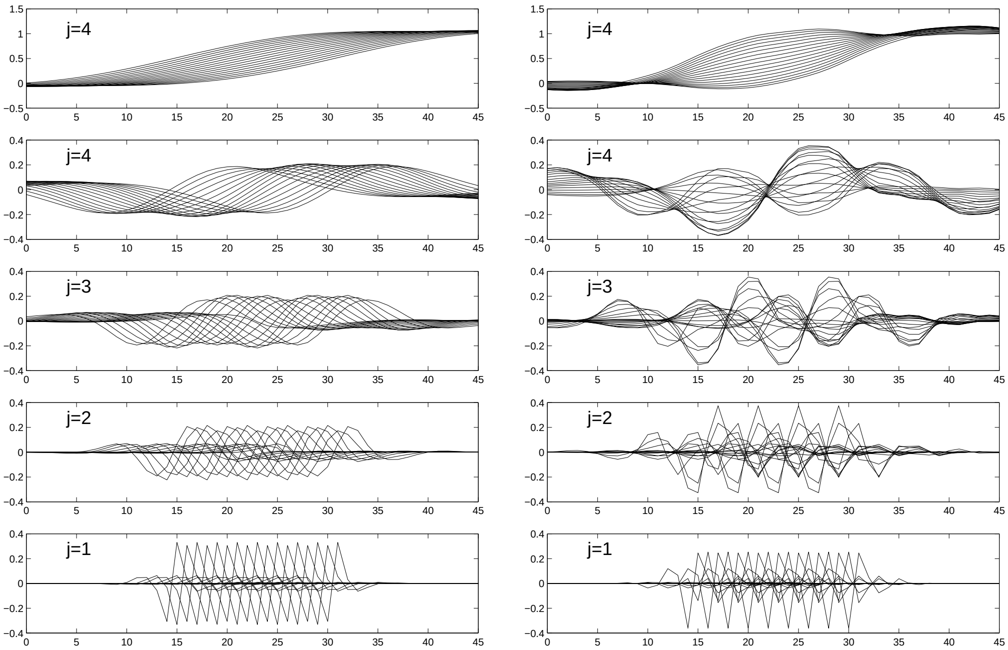

Figure 6: Reconstruction of $u(n-n_0)$ from coefficients at level j only. Left: The decomposition uses the wavelet tight frame illustrated in Figure 4. Right: The decomposition uses Daubechies' orthonormal basis D_5 ($h_0 = 10$). The tight frame is less shift-sensitive than the orthonormal basis.

REFERENCES

[1] S. A. Benno and J. M. F. Moura. Scaling functions robust to translations. *IEEE Trans. on Signal Processing*, 46(12):3269–3281, December 1998.

[2] C. Chui and W. He. Compactly supported tight frames associated with refinable functions. *Applied and Computational Harmonic Analysis*, 8(3):293–319, May 2000.

[3] D. Cox, J. Little, and D. O'Shea. *Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra.* Springer-Verlag, 1991.

[4] I. Daubechies. *Ten Lectures On Wavelets.* SIAM, 1992.

[5] J.-C. Faugère, F. M. de Saint-Martin, and F. Rouillier. Design of regular nonseparable bidimensional wavelets using Gröbner basis techniques. *IEEE Trans. on Signal Processing*, 46(4):845–856, January 1998.

[6] G.-M. Greuel, G. Pfister, and H. Schönemann. Singular Reference Manual. In *Reports On Computer Algebra*, number 12. Centre for Computer Algebra, University of Kaiserslautern, May 1997. *http://www.singular.uni-kl.de/.*

[7] O. Herrmann. On the approximation problem in nonrecursive digital filter design. *IEEE Trans. on Circuit Theory*, 18(3):411–413, May 1971. Also in [12].

[8] N. G. Kingsbury. Image processing with complex wavelets. *Phil. Trans. Royal Society London A*, September 1999.

[9] N. G. Kingsbury. Shift invariant properties of the dual-tree complex wavelet transform. In *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing (ICASSP)*, Phoenix, March 16-19 1999.

[10] J. Lebrun and M. Vetterli. High order balanced multiwavelets: Theory, factorization and design. Preprint, 1998.

[11] H. Park, T. Kalker, and M. Vetterli. Groebner bases and multidimensional FIR multirate systems. *Journal of Multidimensional Systems and Signal Processing*, 8:11–30, 1997.

[12] L. R. Rabiner and C. M. Rader, editors. *Digital Signal Processing.* IEEE Press, 1972.

[13] A. Ron and Z. Shen. Affine systems in $L_2(\mathbb{R}^d)$: the analysis of the analysis operator. *Journal Funct. Anal.*, 148:408–447, 1997.

[14] A. Ron and Z. Shen. Construction of compactly supported affine frames in $L_2(\mathbb{R}^d)$. In K. S. Lau, editor, *Advances in Wavelets.* Springer Verlag, 1998.

[15] I. W. Selesnick. Interpolating multiwavelet bases and the sampling theorem. *IEEE Trans. on Signal Processing*, 47(6):1615–1621, June 1999.

[16] I. W. Selesnick. Balanced multiwavelet bases based on symmetric FIR filters. *IEEE Trans. on Signal Processing*, 48(1):184–191, January 2000.

[17] I. W. Selesnick and C. S. Burrus. Maximally flat lowpass FIR filters with reduced delay. *IEEE Trans. on Circuits and Systems II*, 45(1):53–68, January 1998.