THE PENDULUM: A TIME-FREQUENCY MODEL

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ABSTRACT

We show that time-frequency analysis can be effectively used to study nonlinear differential equations. We study the pendulum in the nonlinear region and propose a new model for the oscillations. We show that when there is damping there exists a fundamental frequency that is time-varying. We explain why that should be so and an explicit method is derived for the calculation of this fundamental time-varying frequency. We further develop a method to obtain the time-varying amplitude. We verify our results by comparing with the numerically obtained solution of the pendulum. We argue that our model allows a deeper understanding of the pendulum oscillation in the highly nonlinear region.

1. INTRODUCTION

Nonlinear dynamical problems have been studied for over 300 years because they immediately arise from the simplest application of Newton's second law.[1] Certainly, the simplest, oldest, and most venerable nonlinear problem is the simple pendulum, which is governed, in presence of linear damping, by the differential equation [8]

$$\ddot{\theta}(t) + 2\mu\dot{\theta}(t) + k^2 \sin\theta(t) = 0 \tag{1}$$

where $k = \sqrt{g/\ell}$, ℓ being the length of the massless pendulum, g the gravitation constant and $\theta(t)$ the displacement angle. This differential equation has been studied in numerous ways over the years and also has served as the model problem for many approximation techniques that have been developed to study nonlinear differential equations.

The linearized version of this equation

$$\ddot{\theta}(t) + 2\mu\dot{\theta}(t) + k^2\theta(t) = 0 \tag{2}$$

valid for small values of θ, gives the standard harmonic oscillator equation which predicts an oscillation with *constant* frequency given by

$$\omega_c = \sqrt{k^2 - \mu^2} \tag{3}$$

when $\mu < k$ (underdamped oscillation). One therefore suspects that the nonlinear equation will have nonconstant frequency, that is, an instantaneous frequency. It is natural therefore to explore the possibility that the new methods that have recently been developed to study signals whose frequency content is changing in time [3] may be profitably applied to these type of problems. Such methods are called time-frequency analysis. There already has been considerable work in studying these types of systems with time-frequency distributions.

Our interest in applying these methods is two fold. First, we will show that these methods may be used to understand the solution of these nonlinear equations. This will be done by solving the equation numerically and studying the solution in the time-frequency plane. Secondly, we believe that the practical issue of solving these equations can be enhanced by devising new approximation methods that are based on examining the equation in the time-frequency plane.

2. THE PENDULUM OSCILLATION IN THE TIME-FREQUENCY PLANE

We numerically obtained the solution $\theta(t)$ of Eq. (1), and then calculated a time-frequency distribution. We choose to work in a high nonlinear region. We start with a large initial angle $\theta_0 = 0.95\pi$ and also impose a low damping $\mu = 0.01$, to obtain a slow decreasing oscillation. We obtained numerically the solution for the time interval $0 \leq t \leq 20$, which gives for a final angle $\theta(20) \approx \frac{2}{3}\pi$, which is still far away from the linear region. Also, we start the motion by taking $\dot{\theta}(0) = 0$. In Fig. 1 we show $\theta(t)$ obtained by numerical integration with the parameters specified above.

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[1] We clear our use of the phrase "non-linear". In the past twenty years or so the phrase has been used in many ways, however we use it in only the classical way. That is, in the sense that the governing equation of the system is a non linear equation, typically a nonlinear differential equation.

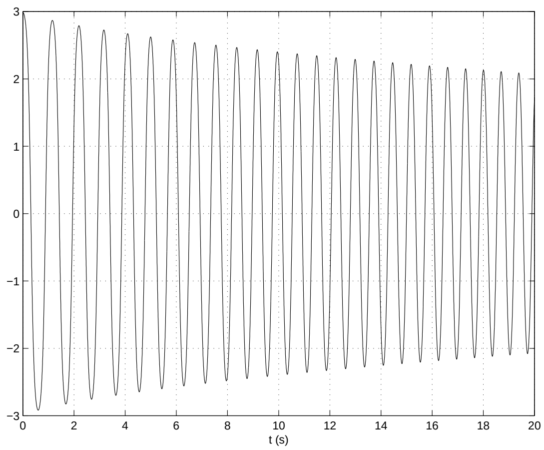

Figure 1: $\theta(t)$ computed with $\mu = 0.01$, $\omega_0 = 4\pi$,$\theta(0) = 0.95\pi$ and $\dot{\theta}(0) = 0$. The final "total" amplitude of oscillation, a concept defined by Eq. (21), is approximetely $A(20) \approx \frac{2}{3}\pi$.

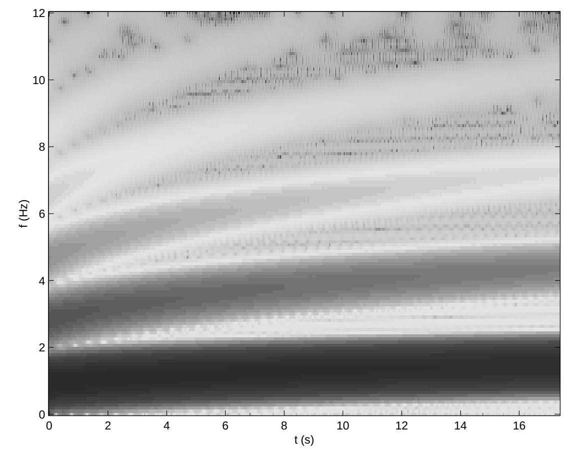

Figure 2: Spectrogram of $\theta(t)$. The fundamental frequency in the frequency interval 0-2 Hz and its harmonics at multiple frequencies are clearly visible. The slope of the harmonics increases with the order.

In Fig. 2 we show the time-frequency plot of $\theta(t)$. We use the spectrogram. [2]

The figure clearly show the presence of a series of time-varying frequencies. In particular it is easily recognizable the presence of a high energy component in the frequency interval $f = 0 - 2$ Hz, plus a set of five components with decreasing energy and increasing slope.

It is instructive to obtain the time-frequency plot of $\dot{\theta}(t)$. The reason is that there is a relationship between the time-frequency plot of $\theta(t)$ and $\dot{\theta}(t)$ and the latter one sometimes shows more clearly the main characteristics of the signal. In Fig. 3 we present the time-frequency plot of $\dot{\theta}(t)$. One notices that the higher harmonics are more clearly visible in Fig. 3, than in the representation of $\theta(t)$. This can be understood in an intuitive way, considering the fact that the relation between the power spectrum of $\dot{\theta}(t)$ and $\theta(t)$ is $S_{\dot{\theta}}(\omega) = \omega^2 S_\theta(\omega)$, and hence the higher frequencies are enhanced in the Fourier spectrum by the multiplicative factor ω^2. This amplification effect is also true for time-frequency representations.[3]

These figures show that the signal is multicomponent [4] and this will allow us to construct a model for these oscillations. We will call the "fundamental frequency", $\omega_0(t)$, the highest energy component, and "higher harmonics" the other components, which will be indicated by $\omega_1(t), \ldots, \omega_4(t)$.

3. HEURISTIC EXPLANATION AND A PROPOSED MODEL OF THE OSCILLATIONS

What the time-frequency plots show is that we have a series of time-varying frequencies. On that basis we will propose a specific model for the oscillations. But first we give a simple heuristic argument as to why there should be these time-varying frequencies. Consider first the pendulum with no damping, that is with $\mu = 0$ [5, 8]

$$\ddot{\theta}(t) + k^2 \sin\theta(t) = 0 \tag{4}$$

It is well know that the solution is periodic and hence can be expanded in a Fourier series. The power spectrum is hence generally made by a set of infinite frequencies, with a fundamental frequency and higher harmonics. It is a classical result that the fundamental frequency is dependent on the maximum angle of deflection. It is given by

$$\omega_0 = \frac{2\pi}{T(A_M))} \tag{5}$$

where $T(A_M)$ is the period which is a function of the maximum amplitude deflection, that here we call A_M. The explicit expression for the period is well known and is given in Section 4. It is important to emphasize that $T(A_M)$ is a constant and hence so is ω_0.

Now, when damping is introduced in the system, the system becomes non conservative, and the maximum deflection changes and therefore the period changes and hence also so does the "fundamental" frequency.

[2] The choice of computing a spectrogram has been made with the objective of representing as many harmonics as possible. It is well known that other time-frequency distributions show a better localization, but the presence of many harmonics generates a big number of interference terms that hide the presence of the harmonics themselves. Hence, despite its poor localization property, the spectrogram with its intensive filtering of the interference terms, is a good tool for a qualitative analysis of the multicomponent behavior of a nonlinear oscillation.

[3] We point out the aliasing produced by the fourth and fifth harmonics, especially in Fig. 3.

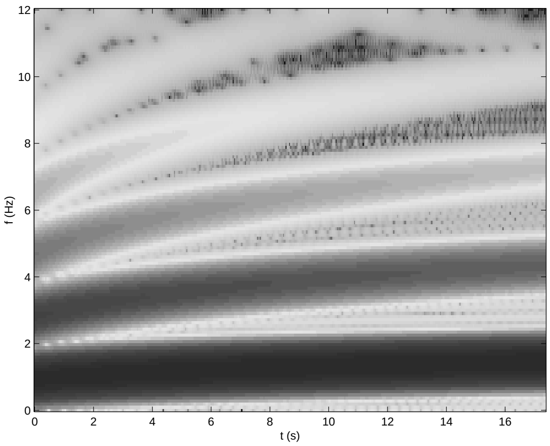

Figure 3: Spectrogram of $\dot{\theta}(t)$. Notice how the higher harmonics can be better identified than in Fig. 2. The fourth and fifth harmonics clearly present an aliasing effect.

We now introduce notation to take this into account. We shall call $A(t)$ the time-varying amplitude of oscillation and we write the period as $T(A(t))$ and the time-varying "fundamental" frequency as

$$\omega_0(t) = \frac{2\pi}{T(A(t))} \tag{6}$$

It is a goal of the paper to model the signal and obtain a method to calculate this fundamental time-varying frequency.

Now if the fundamental frequency changes with time, one expects that also the higher harmonics change, and this is precisely what the time-frequency representations in Fig. (2) and (3) shows. Hence it is plausible to model the oscillations as a multicomponent signal

$$\begin{aligned}\theta(t) &= \sum_{k=0}^{\infty} a_k(t) e^{j\varphi_k(t)} \\ &= \sum_{k=0}^{\infty} a_k(t) e^{j(k+1)\varphi_0(t)}\end{aligned} \tag{7}$$

This model is made of complex functions since that is easier to work with for the usual reasons.

Based on this model the fundamental frequency is

$$\omega_0(t) = \frac{d\varphi_0(t)}{dt} \tag{8}$$

It is interesting to notice that as $t \to \infty$ then $\theta \to 0$ and $\sin(\theta) \approx \theta$ with increasing accuracy in (1). This means that $\varphi_0(t) \to \omega_c t$, where ω_c is defined in (3), and is the frequency of oscillation of the harmonic oscillator. We have made further studies to verify this model and those results are presented in the Appendix. Crucial to our considerations is to obtain the fundamental frequency which is done in the next section. We also point out that most of the energy of the oscillation, even at a relatively high nonlinearity, is concentrated in the fundamental frequency term. This will be discussed further in section 5.

4. THE TIME-VARYING FUNDAMENTAL FREQUENCY

We now show how to obtain the time-varying fundamental frequency. The main idea is to consider the system as locally conservative, that means that if we study Eq. (1) in the time interval $\mathrm{I} = (t_0, t_0 + \Delta t)$, if Δ is small we can consider almost zero the energy lost by the system, and hence consider it as (locally) conservative. As a consequence of this assumption, we consider the total amplitude of oscillation $A(t)$ to be constant in I, at the value that the amplitude has at time t_0. Equivalently we can say that in the time interval I we consider the differential problem (1) to be

$$\begin{aligned}\ddot{\theta}(t) + k^2 \sin(\theta(t)) &= 0, \\ \theta(t_0) &= A(t_0) \\ \dot{\theta}(t_0) &= 0\end{aligned} \tag{9}$$

It is well known that for conservative systems the period is constant and can be written in terms of Jacobi integrals. For this case the period is given by [5]

$$T(z) = \frac{2\pi}{k}\left[1 + \left(\frac{1}{2}\right)^2 z^2 + \left(\frac{1\times 3}{2\times 4}\right)^2 z^4 + \dots\right] \tag{10}$$

where

$$z = \sin(A/2) \tag{11}$$

This is an exact solution for the conservative case.

Considering our discussion above it is therefore natural to take for the non conservative case

$$T(z(t)) = \frac{2\pi}{k}\left[1 + \left(\frac{1}{2}\right)^2 z^2(t) + \left(\frac{1\times 3}{2\times 4}\right)^2 z^4(t) + \dots\right] \tag{12}$$

where now

$$z(t) = \sin(A(t)/2) \tag{13}$$

the fundamental frequency is therefore

$$\begin{aligned}\omega_0(t) &= \frac{2\pi}{T(z(t))} \\ &= k\left[1 + \left(\frac{1}{2}\right)^2 z^2(t) + \left(\frac{1\times 3}{2\times 4}\right)^2 z^4(t) + \dots\right]^{-1}\end{aligned} \tag{14}$$

Hence we have formulated an approximation of the fundamental frequency $\omega_0(t)$ of the pendulum that depends on the total amplitude of oscillation $A(t)$. The approach of considering the system as locally conservative has already been used by one of the authors [6, 7]. In those works $\omega_0(t)$ was expanded with respect to its instantaneous amplitude by using perturbation methods [9]. The inverse problem of nonlinear system classification was investigated, and a low order perturbation expansion proved to be enough to reach the objective. Here, on the contrary, we are interested in the direct analysis of the oscillation in a highly nonlinear region, and we aim at developing an easy way to expand the fundamental frequency with high accuracy. A comparison between the method exposed in this paper and perturbation techniques will be presented later.

4.1. Evaluation of the Amplitude

We now discuss how to evaluate $A(t)$. In Eq. (1), we multiply by $\dot{\theta}(t)$ and integrate with respect to time to get the associate energy equation [8]

$$E_k(t) + E_p(t) + E_d(t) = 0 \tag{15}$$

where

$$E_k(\dot{\theta}(t)) = \frac{1}{2}\dot{\theta}^2(t) \tag{16}$$

$$E_p(\theta(t)) = -k^2[\cos(\theta(t)) - \cos(\theta_0)] \tag{17}$$

$$E_d(t) = 2\mu \int_0^t \dot{\theta}^2(t')\, dt' \tag{18}$$

$E_k(t)$ being the kinetic energy, $E_p(t)$ the potential energy and $E_d(t)$ the dissipated energy. The initial kinetic energy is $E_k(\dot{\theta}(0)) = 0$ because we chose $\dot{\theta}(0) = 0$. The instantaneous energy of the mass m is

$$E_m(t) = E_k(\dot{\theta}(t)) + E_p(\theta(t)) \tag{19}$$

At time t the total amplitude $A(t)$ is equal to the maximum angle θ_M that the mass can reach oscillating when no damping is present, as we hypothesized in the local model (9)

$$E_p(\theta_M) = E_m(t) \tag{20}$$

Substituting we readily obtain

$$A(t) = \theta_M = \arccos\left[-\frac{E_m(t)}{k^2} + \cos\theta_0\right] \tag{21}$$

4.2. Numerical Comparison

Is the expression given by Eq. (21) correct? To ascertain that we computed it and plotted it on the same graph as with the oscillation. This is shown in Fig. 4. The comparison is excellent.

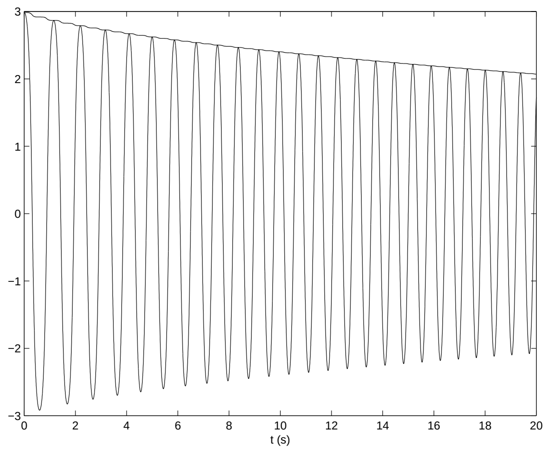

Figure 4: The total amplitude of oscillation $A(t)$, computed applying Eq. (21), with the same parameters used in Fig. (1). The amplitude is compared against the oscillation angle $\theta(t)$, and it well represents its instantaneous envelope.

5. A SIMPLE MODEL OF THE PENDULUM

We have now obtained an estimate of the time-varying fundamental frequency and the time-varying amplitude. Since the time-frequency plot indicates that most of the energy is in the fundamental we would expect that a good representation for the signal is

$$\theta(t) = A(t)e^{j\varphi_0(t)} \tag{22}$$

where

$$\varphi_0(t) = \int_0^t \omega_0(t')\, dt' \tag{23}$$

is the instantaneous phase.

To demonstrate the validity of our model, we compare the estimated fundamental frequency $\omega_0(t)$ and the one predicted by our method, that means Eq. (14) with $A(t)$ computed from Eq. (21). Fig. 5 shows the result of this comparison.[4]

In Fig. 6 we plot the comparison between this prediction and $\theta(t)$ of Fig. 1. The agreement is very good, especially if one considers the highly nonlinear region in which the system is oscillating. The quality of the approximation gives and idea on how it is important the fundamental frequency. Even under highly nonlinear behavior, in fact, the energy of the oscillation still exhibits a good concentration in the fundamental frequency.

[4]We used a number of numerical techniques to obtain numerically the instantaneous frequency.

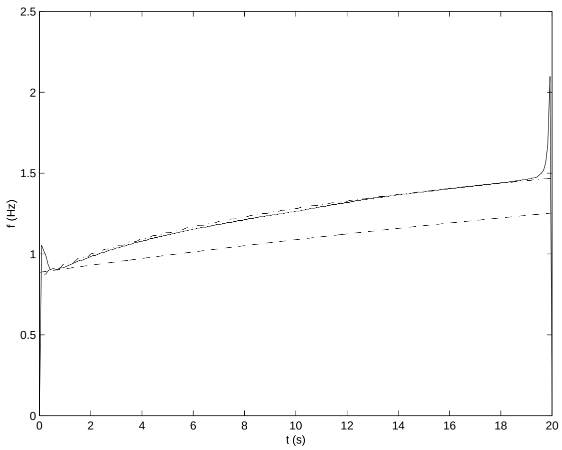

Figure 5: Comparison between the estimated fundamental frequency $\omega_0(t)$ (solid line), the fundamental predicted by Eq. (14) (dashdotted line) and the one predicted by a low order perturbation expansion (dashed line). Notice the quality of the approximation that improves for $t \to \infty$.

6. CONCLUSION

We have studied the nonlinear oscillations produced by a simple pendulum with damping using time-frequency techniques. These tools prove that a nonlinear oscillation has in general a time-varying spectrum, made by a fundamental frequency and a related set of higher harmonics, all time-varying. Driven by this observation, we have proposed a multicomponent model for the oscillation, numerically proving the validity of the choice.

We have also proposed a new model for the fundamental frequency, that allows a better understanding of the mechanism involved in a nonlinear oscillation. Moreover the model allows to approximate the fundamental frequency in the highly nonlinear regions.

7. APPENDIX: NUMERICAL ESTIMATION OF THE FUNDAMENTAL TIME-VARYING FREQUENCY

In Fig. (7), we plot the estimation of $\omega_0(t)$ from both the Wigner distribution [3] of $\theta(t)$ and $\dot{\theta}(t)$. The estimation method used is the peak detection algorithm, that shows good performances for noise free signals [1]. The figure highlights the fact that either of the two signals can be used with success to perform the estimation of $\omega_0(t)$.

In Fig. (8) the estimation of $\omega_0(t)$ from the Wigner and the Smoothed Pseudo Wigner distribution (SPW) is represented. The estimation method is again the peak detection, and the figure shows that they are quite

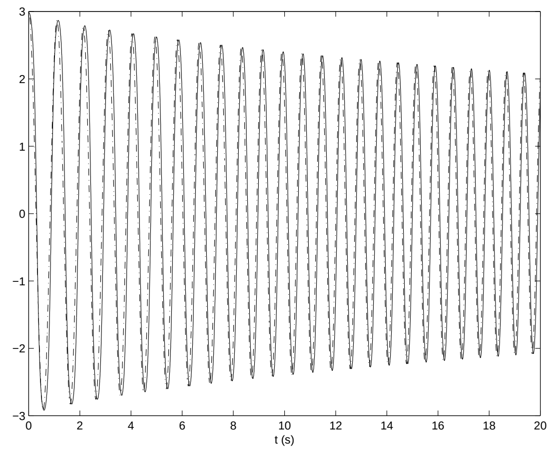

Figure 6: Comparison between $\theta(t)$ numerically integrated from Fig. 1 (solid line) and the one predicted by Eq. (22) (dashdotted line).

the same, although a certain error is present in the SPW approach [2]. Using the SPW can be useful when $\omega_0(t)$ is strongly time-varying, as happens for examples in systems with bilinear functions [6, 7] that generate nondifferentiable points in the fundamental frequency, and hence high interference terms. The SPW can filter part of these terms, allowing the estimation of the instantaneous frequency with the same approach.

Higher harmonics. In the previous paragraphs we have proposed a model, Eq. (7), of the pendulum oscillation. From the model one has that the instantaneous frequency of the k-th harmonic should be

$$\omega_k(t) = \frac{d\varphi_k(t)}{dt} = (k+1)\frac{d\varphi_0(t)}{dt} = (k+1)\omega_0(t) \tag{24}$$

This means that the slope of the k-th harmonic is

$$\frac{d\omega_k(t)}{dt} = (k+1)\frac{d\omega_0(t)}{dt} \tag{25}$$

and this could be an explanation of why the slope of the harmonics in Fig. 2 and 3 increases with the order k. To now show, numerically, that indeed the higher harmonics are multiples of $\omega_0(t)$, we plot in Fig. 9 a comparison between the estimated fundamental frequency $\omega_0(t)$, the estimation of the second harmonic $\omega_2(t)$ and its predicted value $2 * \omega_0(t)$. As it can be seen, the model can be considered satisfactory, especially as for $t \to \infty$.

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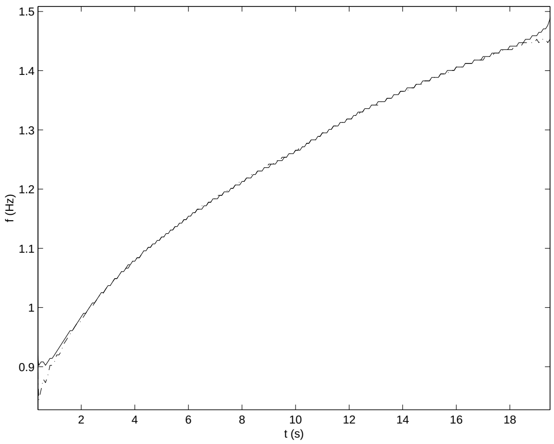

Figure 7: Solid line: $\omega_0(t)$ estimated from the Wigner of $\theta(t)$; dashdotted line: $\omega_0(t)$ estimated from the Wigner of $\dot{\theta}(t)$. The figure shows that both of the signals can be used for the estimation of the fundamental frequency.

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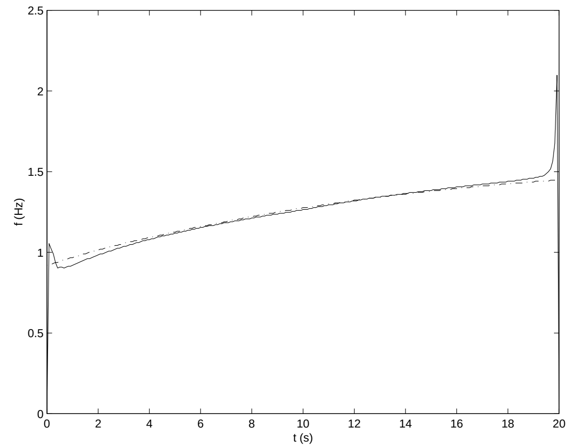

Figure 8: Solid line: $\omega_0(t)$ estimated from the Wigner of $\theta(t)$; dashdotted line: $\omega_0(t)$ estimated from the SPW of $\dot{\theta}(t)$. The SPW estimation shows some biasing especially near the borders of the signal.

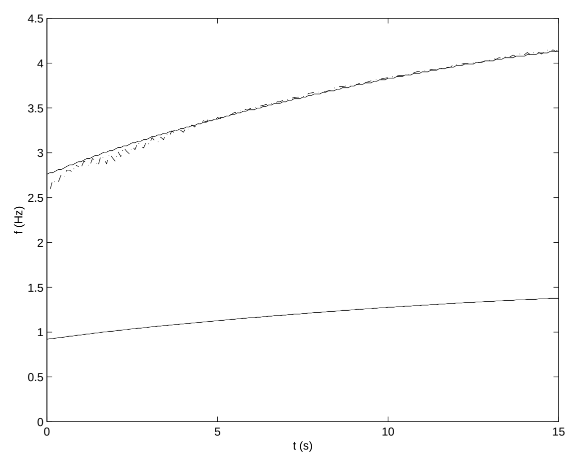

Figure 9: Solid lines: $\omega_0(t)$ and $3 \times \omega_0(t)$; dashdotted line: $\omega_2(t)$. The comparison between $3 \times \omega_0(t)$ and $\omega_2(t)$ confirms that $\omega_2(t)$ can be considered with good approximation a multiple of $\omega_0(t)$. The quality of the approximation improves for $t \to \infty$.